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Phase Transitions Occurring in Models of Neighborhood Racial Segregation

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by

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Extended Abstract

This thesis is organized as two chapters whose contents are closely related yet quite distinct. The first chapter presents a paper "Role of 'Vision' in Neighborhood Racial Segregation: A Variant of the Schelling Segregation Model," authored by myself and Dr. Jaggi, which has been accepted for publication by the journal *Urban Studies* and is currently in press (as of April 2003). This chapter introduces the well-known Schelling model of neighborhood segregation, outlines the sociopolitical motivation for our work, and presents the key results that we believe are of interest to social scientists. Chapter two, which ought to be of greater interest to the physics community, presents the results of our investigations into the parallels between the Schelling model and critical phenomena.

Our primary extension of the Schelling model was to include social agents who can authentically 'see' their neighbors up to a distance R, called 'vision'. By exploring the consequences of systematically varying R, we have developed an understanding of how vision interacts with racial preferences and minority concentrations and leads to novel, complex segregation behavior. We have discovered three regimes: an *unstable* regime, where societies invariably segregate; a *stable* regime, where integrated societies remain stable; and an *intermediate* regime where a complex behavior is observed.

Since the primary audience of *Urban Studies* consists of sociologists and economists, we have not elaborated in the first chapter upon the phase transition which was strongly suggested by the "complex behavior" in the *intermediate* regime. The purpose of chapter two then, is to elucidate these additional physically interesting aspects of our model. Melting is a textbook example of first order (discontinuous) phase transitions. These are marked by two central features: a *sharp* temperature at which the transition occurs, and the coexistence of the two phases at that melting point. One can study the first-order phase transition that ice undergoes when melting into water by observing the ice while continuously raising its temperature. However, if you were only able to view the system at certain discrete temperatures, you would only see a either a piece of ice or a puddle of water during each observation. Thus in order to study the potential phase transition occurring in our model, we must be able to control the governing parameters continuously. However, in our original 'discrete' model, R measures how far an agent sees from its own home as an integer number of houses. Since we can only assign discrete values to R, it is meaningless to speak of a phase transition occurring as a function of this variable.

To overcome the limitations of our first model, we introduce a *continuous* model in chapter two where the range of vision (denoted R_2 for notational clarity) can be varied continuously. This model uses a utility function that assigns greater weight to neighbors nearer an evaluating agent. The function used to model this decrease in utility contribution with distance is an exponentially decaying curve. We control the steepness of this curve (and thereby control the agents' vision) using R_2. Since R_2 can be set to

equal any positive real number, we can indeed study the possible phase transition in our simulations' behavior as the function of a continuous variable.

Additionally, the continuous model demonstrates the robustness of the sociologically relevant conclusions drawn in chapter one. Our continuous model, a generalization of a model developed by Wasserman and Yohe (2001), is in fact more realistic than our first model. In particular, we were pleased to discover the same three behavioral regimes and all associated trends in both our discrete model and our continuous model. This confirms that our original results were robust and not merely algorithmic artifacts related to the specific treatment of vision used in our discrete model.

CHAPTER ONE:

Role of 'Vision' in Neighborhood Racial Segregation: A Variant of the Schelling Segregation Model

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Abstract

We have extended the Schelling model of neighborhood racial segregation to include agents who can authentically 'see' their neighbors up to a distance R, called 'vision'. By exploring the consequences of systematically varying R, we have developed an understanding of how vision interacts with racial preferences and minority concentrations and leads to novel, complex segregation behavior. We have discovered three regimes: an *unstable* regime, where societies invariably segregate; a *stable* regime, where integrated societies remain stable; and an *intermediate* regime where a complex behavior is observed. We present detailed results for the symmetric case (which maximizes conflict), where equal numbers of agents of two races occupy the same cityscape. We briefly indicate the policy implications of these simulations.

I. Introduction:

Neither the passage of time—thirty five years since the Civil Rights movement, nor of laws --the Fair Housing Act of 1968 or the Equal Credit Opportunity Act of 1974, for example, have eliminated neighborhood racial segregation in these United States. It is true (Farley and Frey, 1991); (Farley et al., 1993); (Thernstrom and Thernstrom, 1997) that there has been some decrease in the intensity of racial segregation in small and medium cities with relatively small black populations. But, the levels of residential segregation of African- Americans in the major metropolitan areas continue to be pervasive and persistent (Massey and Denton, 1987); (Massey and Denton, 1993).[1]

The scholarly analyses of the explanatory factors possibly influencing this sociological condition have been quite exhaustive. Differences in income, housing affordability, location of employers and businesses, crime, job opportunities, neighborhood racial preferences, racial steering by real-estate professionals, lending practices of financial institutions, and a host of other factors have been invoked as likely causes (Carr, 1999). The status of the literature on this explanatory enterprise is represented, quite fairly in our opinion, by the following quote (Clark, 1991), " In the debate about the relative role of these forces, the consensus is that the patterns of separation have a multifaceted explanation: Among the explanatory factors, neighborhood composition preferences have been singled out as a critical variable both by economists, who view preferences from the perspective of consumer behavior theory, and by geographers and sociologists, who use preferences and expectations as elements of models of residential choice within cities and neighborhoods." This study further explores the consequences of individual preferences on neighborhood segregation.

In two pioneering papers, Schelling (1969; 1971) introduced a model system of two distinguishable types of agents with discriminatory individual preferences for certain neighborhood compositions, and then quantitatively explored the dynamics of this model system.[2] In effect, this work was perhaps the first concerted and systematic application of what is now called the agent-based modeling approach to sociological systems. The towering status of Schelling's legacy in this area is such that, before writing yet another paper on the subject, perhaps we ought to pause and ask ourselves, "didn't Schelling already say that?"[3] Our goal in this introduction is simply to locate our specific points of departure from Schelling's model and from his overall insight. For the limited purpose of framing our own work therefore, and because we see our work as an extension of the

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[1] Massey and Denton coined a term, "hypersegregation" to describe this multi-dimensional, pervasive and persistent segregation.

[2] While the original journal article (Schelling 1971) is a tour-de-force in mathematical sociology, the less mathematically inclined among us can get the same insights from later essays written for broader audiences (Schelling 1971a, 1978).

[3] This delightfully worded question was first asked (Martin 1999) in a slightly different context of contingency and specificity of formal models in the area of international security studies, " I think of this as the generic "didn't Schelling already say that?" question."

Schelling model, we permit ourselves the following brief, though admittedly narrow characterization of Schelling's rather deep insights.

His insight in this subject has frequently been described by scholars by comments of the following kind: "Quite minor differences" in individual preferences lead to "aggregate results that are strikingly different" (Clark, 1991). Schelling's seminal work has since been supplemented by numerous scholars who, for example, conduct "... a Test of the Schelling Segregation Model" and "... confirm the view that stable integrated equilibria are unlikely" (Clark, 1991). Another author (Krugman, 1996) concludes, "Guess what: even though individuals are tolerant enough to accept an integrated pattern, they end up with more or less total segregation." A very recent paper (Wasserman and Yohe, 2001) "examined the robustness of his conclusions in two slightly more realistic environments" and found "strong support" for Schelling's conclusions. They write, "The second case expanded residents' vision ... The segregation in the equilibrium neighborhood was, in fact, even more obvious than before." Other scholars (Epstein and Axtell, 1996) have written of Schelling's work, "He found that even quite color-blind preferences produced quite segregated neighborhoods". In the present work, we qualify these claims and make explicit the contingencies surrounding assertions of this kind.

This way of framing the implications of the Schelling-model, and the corresponding tone of surprise about the apparent disconnect between agent-intent and consequences, have also found their way into influential popular writings on the subject. A good example of a commentary on the Schelling model is found in a well-researched recent essay (Rauch, 2002) in the literary magazine *The Atlantic Monthly*, where the essayist writes, "In the simulation I've just described, each agent seeks only two neighbors of its own color. That is, these "people" would all be perfectly happy in an integrated neighborhood, half-red, half-blue. If they were real, they might well swear that they valued diversity. The realization that their individual preferences lead to a collective outcome *indistinguishable* (italics ours) from thoroughgoing racism might surprise them no less than it surprised me and, many years ago, Thomas Schelling." And a particularly bad example-- not only because of its failure to understand the nuanced nature of Schelling's work but also because of its illogical concatenation of Schelling's observations with racist ideology—can be found in an article titled, "Racial Segregation is an Inevitability". On the web-site of a British Aryan Unity group, one author (Ormerod, 2002) has used-- abused in our view[4] -- Schelling's work to pronounce, "But this does not undermine Schelling's central insight. Marked segregation can arise from only rather mild individual preferences. The main problem faced by British society is neither one of racist attitudes nor of residential segregation. Rather, it is the ideology of multi-culturalism." In the present work, we provide evidence against such claims of inevitability of segregation in Schelling-like models.

[4] We wish to assert that this sense of inevitability of racial segregation, whether of admiring scholars or of politically motivated groups, receives no support from Schelling's own writings, which are quite circumspect and display an awareness of the contingencies of his celebrated model.

There is robust empirical evidence (Farley and Frey, 1991); (Thernstrom and Thernstrom, 1997) that there has been some significant decrease in the intensity of racial segregation in small and medium cities in the United States. Is Schelling's model simply incompatible with these empirical findings? We were interested in revisiting Schelling's model to see if one forgot, as it were, to look in some areas where there might be evidence against this notion of inevitability of segregation.

It is important to acknowledge that there have been other studies that have extended agent-based models of racial segregation and racial transition of neighborhoods, typically by trying to make the model more complex and realistic. One early, but important example of such studies is a policy simulation (Vandell and Harrison, 1978) which postulates a third party -- the real estate speculator-- who stands in for the array of institutional intermediaries that play, in reality, a major role in the racial transitions in neighborhoods. Having acknowledged this class of studies, we point out that the thrust of the present work is quite different. It argues for the inclusion of an *essential* new feature of the agents, their vision, in this class of models. Without 'vision', we believe, these models will remain fundamentally incomplete.

II. Scope of This Study:

The thrust of our work is then to revisit Schelling's model and examine it closely, comprehensively and systematically with respect to one particular parameter we believe to be quite significant, viz. the *range of vision, R* of the agent. We do not claim to be the first to have studied the effect of vision in this context. After finishing our simulations and during the writing phase, we became aware of two very recent studies, one unpublished (Sander et al., 2000) and the other published (Wasserman and Yohe, 2001), which are related to our work. Both these studies recognize the importance of 'seeing' other agents who are at some distance from the agent who is making the choice whether and where to relocate, and the authors include this effect in different ways in their analyses.

One study (Wasserman and Yohe, 2001) introduces a utility function that decreases exponentially with the distance from the agent making the decision, as a way to include the effects of racial composition away from the agent. They (Wasserman and Yohe, 2001) conduct computer simulations in a portion of the parameter space and conclude with a "strong support" for Schelling's claim of segregation. They report, "The second case expanded residents' vision … The segregation in the equilibrium neighborhood was, in fact, even more obvious than before. … This result suggests that segregation is positively correlated to the vision parameter—an observation that is also consistent with Schelling's hypothesis."

In the other study (Sander et al., 2000), the neighborhood of approximately 2500 homes is divided into 25 fixed tracts each containing 100 cells. The utility function of the agent depends upon, apart from the Moore neighborhood (see fig 1), upon the racial composition in the tract in which the agent is currently located and the tract which

contains the trial site where the agent is considering to move. They can vary the relative weights of the Moore neighborhood and of the tract. They find that as the weight of the extended tract is increased from 0 to 1, the dissimilarity index increases from about 0.4 to 0.8. This conclusion is similar to that of Wasserman and Yohe (2001), viz. increasing the vision makes segregation worse. In our study, we introduce a different way of including vision and study its effects by systematically varying the range of vision. We discover that the effect of increased vision is, in fact, more complex and interesting than implied by these very recent studies.

In the standard Schelling model of a two-race 'artificial society' (Epstein and Axtell, 1996), agents are characterized by a parameter p, which is a measure of their preference for agents of their own kind. Agents evaluate the racial composition of their *immediate* neighborhoods and compare the composition with their own value of racial preference, p, to determine whether they will attempt to relocate elsewhere. Our variant differs from this in that the agents in fact *"see"* their neighborhood[5] up to a certain 'distance' R from their own home while evaluating their decisions to relocate.

We find, depending upon the values of p and R, that the system evolves in one of two possible modes. In one region of the parameter space, the system displays the familiar (Schelling, 1969; 1971; 1971a; 1978); (Epstein and Axtell, 1996); (Sander et al., 2000); (Wasserman and Yohe, 2001) mode where initially integrated communities are unstable and quickly resegregate. We call this the *unstable* regime. But we have discovered that there is a large region of the parameter space (p,R), particularly for moderate values of R ($2 \leq R \leq 7$), where integrated communities remain stable for arbitrarily long times. We call this the *stable* regime.

It is important to note that what we have called the stable regime does not correspond to some unrealistic, Gandhian levels of racial preferences/tolerances of the agents. Once the range of vision R is expanded from myopic levels (say R=1) to rather modest levels (say R= 3 to 5), non-segregated stable communities are found to be fully consistent with non-zero and quite substantive values of p. If this insight were to diffuse into the collective consciousness of policy makers and of the general populace, it could help generate an optimistic outlook for the future of neighborhood integration.

[5] In its standard demographic usage, the term 'neighborhood' evokes a region with fixed boundaries of a specified size. We use the term 'cityscape' for this and reserve the term 'neighborhood' to denote the agent-specific and variable subset of homes within a certain distance of the instantaneous location of the agent.

III. Details of the Extended Schelling Model:

One of the central issues in agent-based modeling approaches to sociological problems is the level of complexity that is appropriate: it needs to be realistic enough to lead to acceptable level of congruence with empirical findings without being so complex as to provide no insight. It is worth paying heed to what Schelling himself had to say on the subject very recently (Holden, 1996), "Thomas Schelling points out that the more complex and therefore realistic the model becomes, the more difficult it will be to discern cause-effect relationships -- just like real life." Since one is interested in maintaining, if possible, a sense of cause-effect relationships, we are cautious in this study and extend his model by including just one additional parameter, i.e. range of vision. In particular, we intentionally restrict it to agents of only two distinct kinds. It is perhaps better to refer to them as 'type one' and 'type two' because the model does not care about the underlying reason for the preference p, which is simply a parameter that can be set to any value between 0 and 1. There are two reasons why we nevertheless permit ourselves to use the racial labels 'white' and 'black' in the rest of this paper. First, in a journal that doesn't accept color pictures, the only reasonable way to construct a graphic depiction of agents of two types *and* vacancies is to use black, white and empty sites. Second, it is only fair to acknowledge that our primary interest is in bringing the issue of range of vision to the discourse on racial segregation.

This notion of range of vision is best introduced by referring to an aspect of the Schelling model that can be a bit confusing at times. For example, Rauch says (2002), "In Schelling's model, unhappy agents, like the modeler himself, could survey the whole scene to find a better situation." This needs to be made more precise. There are two quite distinct kinds of surveying that the agents are capable of. When deciding *where* to move, the agents do survey the whole scene for an available spot, as Rauch states correctly. But when it comes to deciding *whether* to move, the agents scan *only* their immediate neighbors. As if these agents possess a dual vision! The actual situation in the literature is slightly more complicated than this. In all previous studies however, the vision for determining the 'neighborhood composition' is exceedingly myopic, whereas the vision for locating satisfactory sites is unrestricted.

We define the R-neighborhood of an agent as the set of all sites that can be reached by traveling R spaces in any combination of the cardinal directions. In our model, the agent *does, in fact,* evaluate the racial composition in this extended R-neighborhood to determine the satisfaction of his/her racial preferences in deciding whether to look for a site to move to.

Note that the R-neighborhood is not a fixed-boundary neighborhood or a tract of the sort Sander et al. (2000) employed. We let each agent define its own neighborhood, centered about itself. We feel this is a more realistic definition, especially for agents that would otherwise be near the boundary of a tract.

In the spirit of avoiding unnecessary complication, we restrict, at least in the present study, all agents in a given ensemble to have the same range of vision *R*. We treat *R* as a non-dynamic, variable parameter and systematically explore the effect of varying *R* from one simulation to another. Our goal is to develop a qualitative sense of what the consequences are of varying the vision *R*. In future studies, one might wish to consider inhomogeneous models characterized by a range of distributions for *R* even within the same ensemble of agents.

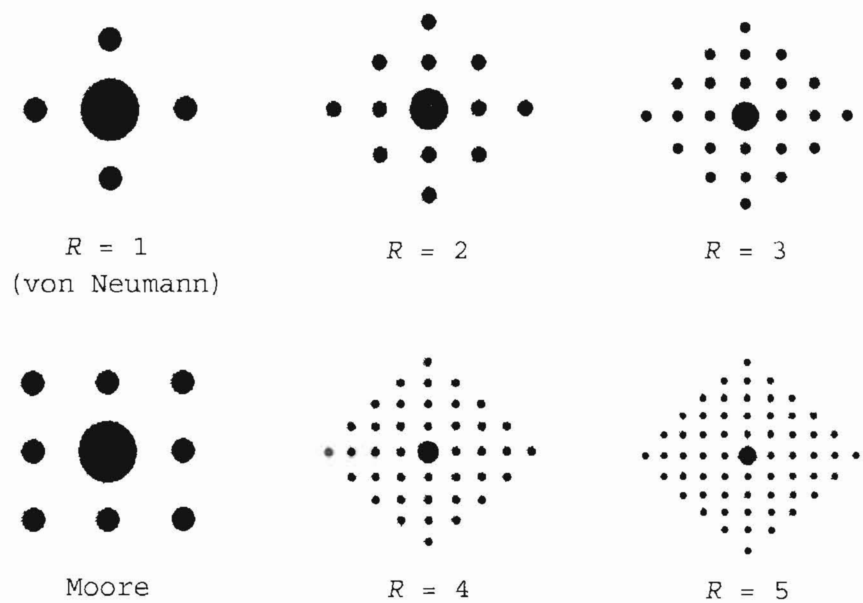

Fig. 1
Neighborhoods specified by Vision
The large dots in these figures represent the agent evaluating its neighborhood, and the smaller dots represent those agents it considers part of its *R*-neighborhood. The Moore neighborhood is also shown for comparison. Roughly speaking, the Moore neighborhood is equivalent to what we might call "$R = 1.5$"

Our computer simulation essentially does four things: it creates a randomly generated artificial society, it repeatedly evaluates the satisfaction of individual agents and if necessary moves them, it creates a graphic display of the society for the user, and it measures the segregation of the neighborhoods.

When the goal is to attempt a direct contact between simulations and demographic data for specific cityscapes, one must obviously use the *known* geography and the *known* boundary conditions. But our central goal in this work is to extract and to present an understanding of how varying *R* affects the nature of segregation, while minimizing computational artifacts. Small size effects and edge effects are the two important artifacts that all modelers must deal with, making a compromise between the desire to use

a large society (which reduces ensemble-to-ensemble fluctuations) and the limits to computational power at one's disposal. Due to the computer memory limitations of his day, Schelling's pioneering work employed 13x16 or 8x8 arrays. However, in contemporary research such as that of Epstein and Axtell (1996) and our own, the chosen geography of societies consists of square NxN arrays, where N=50. Additionally, we use periodic boundary conditions in both dimensions, i.e. the 'east' and 'west' borders of the society 'wrap around' to meet one another, and the 'north' and 'south' borders do the same. So, technically the society is on an edgeless torus (i.e. donut), not on a flat grid. This has the additional advantage, in the present context, of suppressing boundary effects, so that any observed variation in segregation pattern can be reliably attributed to variation in R.

The parameter c represents the concentration of the minority race. Since the central goal of this study is to understand the effect of the range of vision R, we have intentionally kept the model symmetric between the two races. Thus equilibrium configurations for c and $(1-c)$ are identical except that labels for 'black' and 'white' are switched. Therefore c is restricted in the range $(0 < c \leq 0.5)$. An agent's preference p is simply the minimum fraction of agents of its own race it must see in its R-neighborhood to be satisfied, i.e. to have no desire to move. The parameter v $(0 \leq v < 1)$ represents the concentration of vacant homes.[6]

The code randomly picks $(1\text{-v})\ N^2$ sites in the array to be initially occupied by $c(1\text{-}v)\ N^2$ 'blacks' and $(1\text{-}c)(1\text{-}v)\ N^2$ 'whites'. During each iteration of this portion of the program, an agent is selected at random for 'evaluation'. The agent looks around in its R-neighborhood. If the computed ratio of agents of the same race to total agents in its R-neighborhood is greater than or equal to its preference p, the agent is *satisfied*: it does nothing and this iteration is finished. However, if this ratio is lower than p, the agent makes a series of attempts to move.

In attempting to move, the agent randomly selects a vacant site, anywhere in the society, and performs the same kind of evaluation it performed at the site of its current home. If there would be a greater fraction of like neighbors in the R-neighborhood of this 'trial' vacant site, thus potentially increasing its *satisfaction*, the agent moves, leaving its old home vacant. If the move would not increase the *satisfaction* of the agent, it randomly selects a different vacant site to evaluate if it can increase its *satisfaction* by moving. This process is repeated up to vN^2 times, corresponding to each vacant site being evaluated an average of once, before admitting defeat, for the time being, and staying put.

For instance, with a 50x50 society with 90% occupancy, an agent would view at most 250 new homes before either finding a suitable new location or admitting defeat until selected again. It turns out that, in actual simulations, very few agents are forced to

[6] It turns out that R, p, and c are the interesting, essential and dominant independent variables in this model, whereas v affects the segregation peripherally. For this reason, all results reported in this paper correspond to a typical value of v=0.1.

admit defeat unless the society is characterized by very high values of p, the preference for one's own kind.

The 'evaluation' portion of the program is repeated until no more agents wish to move: equilibrium has been reached. Typically, this occurs when everyone is satisfied, but equilibrium societies also occur where agents are dissatisfied but no 'better' locations exist to move to. Again, the latter occurs primarily in high-preference societies.

One might wonder why, in our model, an agent moves to the *first* site it finds with a higher fraction of like neighbors. Wouldn't an agent want to move to a site with the *highest available* fraction? We note that because we have chosen not to include a 'moving cost', this question turns out to be moot. Since agents have no cost of movement, they are able to move freely as many times as they wish. Furthermore, every move increases the fraction of like neighbors for both the agent moving and (for societies with p □ 0.5) for all agents in the mover's old and new R-neighborhoods. So, only 'utility improving' behavior occurs, and in all equilibrium societies that we present in this paper, all agents are maximally satisfied.

Two additional items of detail are worth pointing out. First, the total number of agents remains the same: the model, by design, does not allow mobility of agents completely off the cityscape (that is a different model altogether). Also, simple two-way trades of homes are not incorporated into the model, because they are, apart from being unrealistic, believed to be irrelevant in determining the final equilibrium configurations.

We construct a metric, S, called the "ensemble averaged, von Neumann segregation coefficient at equilibrium" as follows. Each agent looks at its von Neumann neighborhood, calculates the actual fraction of neighbors of like race to the total number of neighbors (this denominator isn't always four since some agents have vacancies in their neighborhood). Then, this fraction is 'scaled' in such a fashion that it yields a contribution between 0 and 1, where 0 corresponds to what one would find in a random initial society of the given concentration c, and 1 represents a total segregation. These scaled values are then averaged over all agents to determine S. This metric is closely related to the dissimilarity index used in the demographic literature, and approaches the same value (0 and 1) in the limits of complete integration and total apartheid respectively.

Formally, S is defined by the following equation,

$$S = \frac{1}{(1-v)N^2}\left[\sum_{j,white}\frac{(f_j - f_w)}{(1-f_w)} + \sum_{k,black}\frac{(f_k - f_b)}{(1-f_b)}\right]$$

where $f_w(c)$ and $f_b(c)$ represent the expected fraction of white or black neighbors respectively in a completely random initial society with concentration c of minorities.

Note that S can, in principle, take on negative values! Imagine a 'checkerboard' society with no vacancies and c=0.5, in which the sites were filled with a perfectly alternating pattern of the two races. In this case, the numerical value of S would actually be -1. In this paper, however, an unbiased random number generator with appropriate probabilities generates the initial configuration of a society. At the beginning of the simulation, the configuration is always integrated: typical values of S in this starting configuration are 0.00 ± 0.02. Since the agents only move to locations with greater fractions of their own kind-- In this rather conservative model, an agent never moves to a site whose R-neighborhood is more diverse than its current site-- S either remains close to zero or increases with time.

IV. Results:

IV (a). The Effect of Varying *R* for the Case of Moderate Preferences:

All results presented in sections III and IV are for the case c=0.5, corresponding to equal numbers of two races trying to occupy the same cityscape. Initially, we explore the case p= 0.5, because it is the prototypical case and has been the focus of much earlier work. For R=1 (see Fig. 2), even though the initial society (left panel) was random and fully integrated, the final, equilibrium society (right panel) is substantially segregated, in agreement with earlier work (Schelling, 1971); (Axtell and Epstein, 1996); (Wasserman and Yohe, 2001); and (Sander et al., 2000). But we point out that the segregation coefficient S is about 0.62, much less than the value of S=1, which characterizes *total* segregation. It is therefore inappropriate to describe this as 'almost complete' segregation. We also wish to emphasize the dendritic nature of the pattern, which is distinct from the two-domain segregation that we present next.

Now, we exhibit results that demonstrate the consequences of expanding the vision of the agents. For the same preference, i.e. p=0.5, but with the agents enabled to 'see' sites up to five spaces away from them, i.e. R=5, the structure of the equilibrium society is strikingly different from the von Neumann case, as evidenced by Fig. 3.

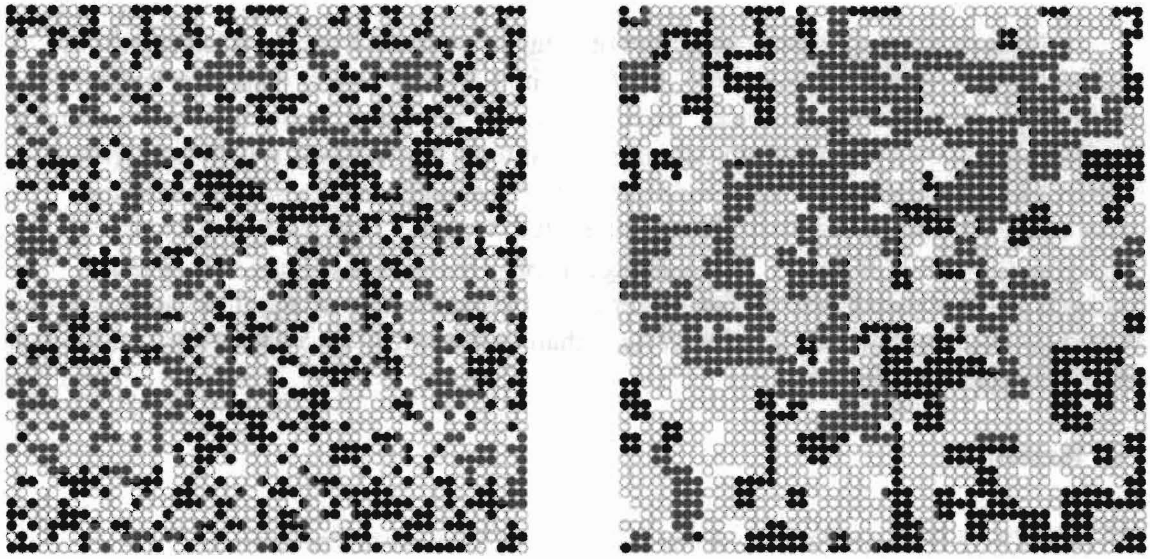

Fig. 2
Initial (left) and Equilibrium (right) Societies for $R = 1, p = 0.5$
'Small-domain' segregation occurs. This is characterized by small, partially interconnected (dendritic) ghettos.

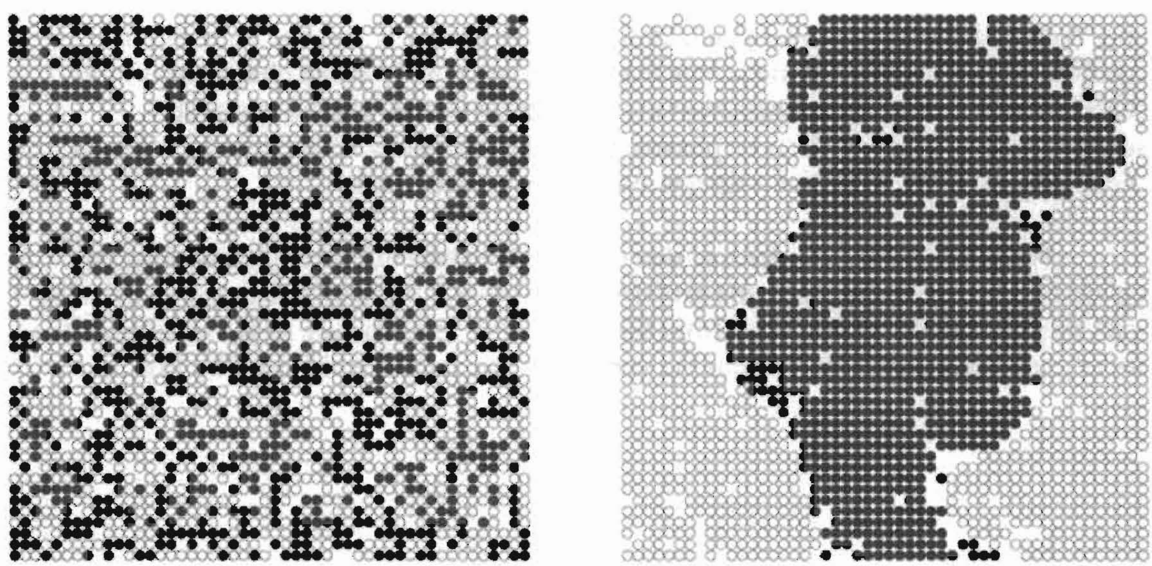

Fig. 3
Initial (left) and equilibrium (right) societies for $R = 5, p = 0.5$
Since agents require half of their neighbors in a larger R-neighborhood (containing 60 sites) to be of their own race, 'large-domain' segregation occurs. In fact, a 'complete' segregation occurs, leading to only two domains (i.e. ghettos) at equilibrium.

Now, the segregation is much "worse" than in the von Neumann case in two aspects. First, the pattern is better described as two isolated domains (or ghettos, if one wants to use the term), quite unlike the dendritic, small-domain segregation for the prototypical case of p=0.5, R=1, presented in Fig. 2. Second, the computed segregation coefficient S for the equilibrium neighborhood in Fig. 3 is approx. 0.97, reflecting the obviously far greater degree of segregation as compared to Fig. 2,which was characterized by S=0.62. Perhaps one should reserve the terms 'complete' or 'total' segregation for such two-domain segregation with S close to 1.0.

Wasserman and Yohe (2001) first noticed that when they expanded their resident's vision (effectively at $p = 0.5$), "even more obvious" segregation occurred. This is similar to the findings of Sander et al. (2000) that as the relative weight of the extended tract (as compared to the Moore neighborhood) is increased from 0 to 1, the dissimilarity index increases from about 0.4 to 0.8. Even though these two studies, (Wasserman and Yohe, 2001) and (Sander et al., 2000), had introduced the effect of distant agents in quite different ways, this result of ours, i.e. increased segregation with increasing vision, is in qualitative agreement with their results.

Wasserman and Yohe's utility function incorporates an agent's desire to be near its own kind and the agent's desire to be far from the other kind, according to the formula:

$$U_j = \sum_{i=0}^{n} 2^{-(d(i)-1)} + \lambda \sum_{k=0}^{n} 2^{-(d(k)-1)}$$

where $d(i) \geq 1$ is the distance of a neighbor of individual j's own race, $d(k) \geq 1$ is the distance of a neighbor of a different race, n is the number of neighbors within a range of vision, and λ is a measure of the agent's attitude towards members of the other race. Negative values of λ correspond to an agent wanting to be far from agents of the other race, but positive values of λ indicate that an agent wants to be near agents of the other race (possibly even more so than agents of its own race if $\lambda > 1$). Wasserman and Yohe always use $\lambda = -1$, so that equal weight is attached to an agent's desire to be near its own kind and its desire to avoid the other kind. An agent will try to move if its utility falls below a certain value, and Wasserman and Yohe present results for simulations in which the threshold is zero. Note that this is qualitatively similar to our moving criteria, except that in our model the effects on utility do not decay with distance from the agent. Thus, Wasserman and Yohe's value of $\lambda = -1$ corresponds to p=0.5 in the language of our simulations: an agent will move if the (weighted or unweighted) fraction of like neighbors equivalently falls below 50%. Incidentally, Wasserman and Yohe also studied the effect of public goods in their work. But our work does not include any such external factors.

At first, this consequence of increased vision can appear a bit surprising. But think about a stage where a cityscape has just begun to segregate. Now if R=1, agents directly adjacent to a ghetto may wish to move. But, if R were 5, agents several sites

away from the ghetto boundary are also likely to be unsatisfied. Thus, for larger R, a greater number of moves will occur in each simulation, and each move increases segregation. It is this 'amplifying' effect of R during the early stages of segregation dynamics that explains the 'worsening' for the case p=0.5.

IV (b). The Effect of Varying *R* in the *Stable* Regime:

We now explore the consequences of varying R for the case where the agents' preferences for their own kind are a bit smaller, but still nonzero. i.e. when societies are a bit more enlightened. In this study, we are not interested in utopian societies, i.e. one in which agents actively seek *more* diverse neighborhoods than their current site. In fact, this is explicitly forbidden in the present model. In this model, agents move only into less diverse neighborhoods, to find more neighbors of their own kind, to satisfy their racial preferences. Nor are we interested in unrealistic agents with a Gandhian, color-blind world-view, where one does not care at all about the typology of one's neighbors: this corresponds to p=0. It is worth emphasizing, if only because many observers and commentators have mischaracterized p=0.5 as "quite color-blind", e.g. (Epstein and Axtell 1996), that the prototypical case of p=0.5 is far from representing color-blind agents: it, in fact, corresponds to an agent who never wants to be in the slightest minority under any circumstance. This agent demands that 50% of its neighbors must be of its own kind, at all times. If not, this agent tries to move, aggressively and repeatedly, until it finds a suitable site or until it has checked out all possibilities.

We could pick a point half way between being a color-blind Gandhian (p=0) and one who moves continually to avoid ever being in the slightest minority under any circumstance (p=0.5): this halfway point would correspond to p=0.25. In order to be more realistic, we first present the results for the case, p=0.3. Fig. 4 shows our results for p=0.3 and p=0.5 for R=1, 3, and 5. Recall that in this paper, an unbiased random number generator with appropriate probabilities always generates the initial configuration of a society, which is always integrated. Therefore, in the rest of the paper, we do not display the initial state of the society. All the panels in Fig. 4 (and in later figures) correspond to the final, equilibrium configurations of the society.

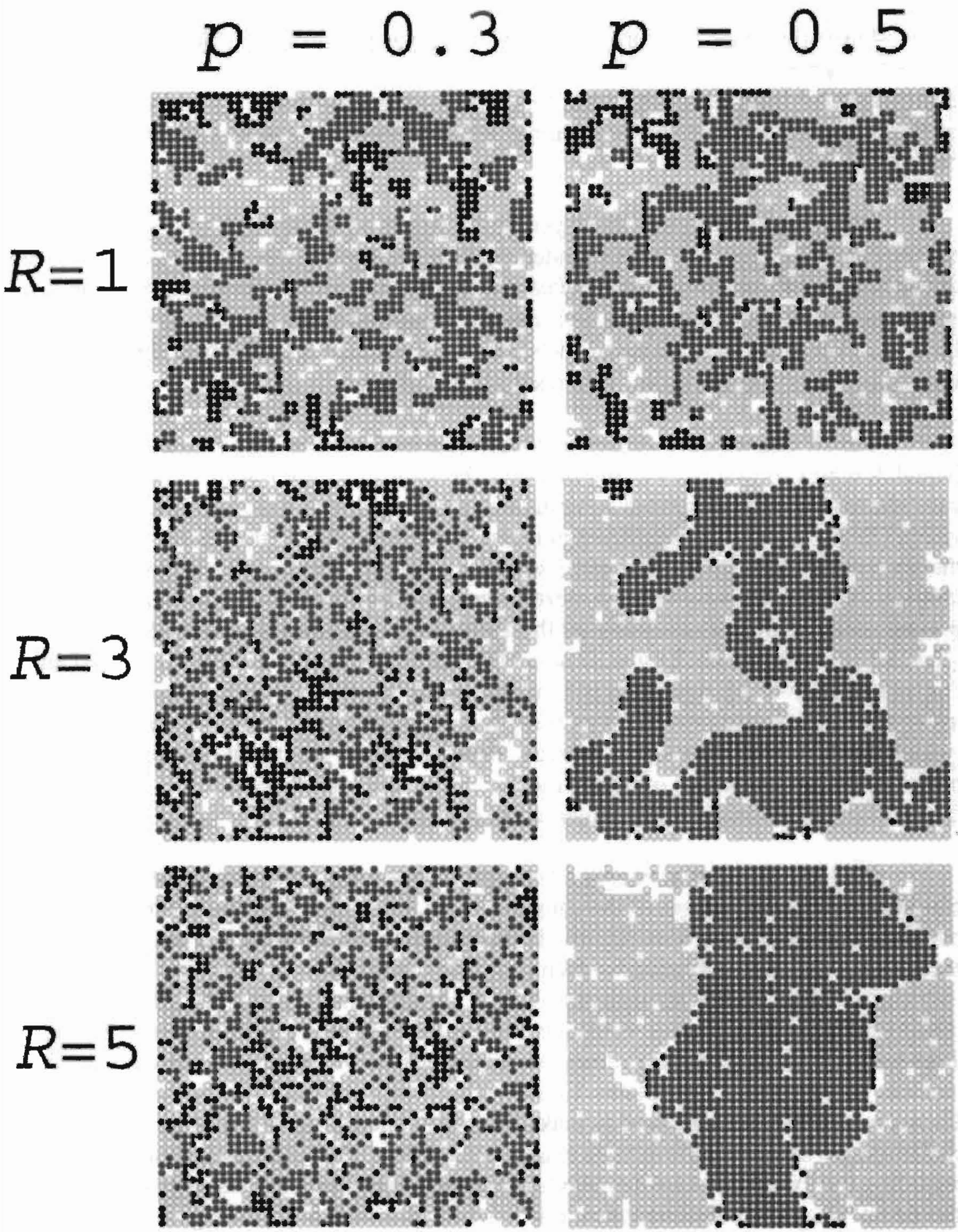

Fig. 4
Equilibrium societies for different values of *R* and *p*
The left column corresponds to the *stable* regime (p=0.3) and the right column corresponds to the *unstable* regime (p=0.5).

For R=1, the equilibrium societies for both cases (p=0.3 and 0.5) appear to have what we have called "small domain" or "dendritic" segregation. In fact, they are barely distinguishable from each other. And for the p=0.5 case, as demonstrated by the panels on the right half of Fig. 4, the "worsening" effect of increased vision is again quite obvious.

But for the p=0.3 case, displayed by the panels on the left half of Fig. 4, it is obvious that as R is increased, the tendency of the society towards segregation is reduced dramatically, and monotonously. Indeed, the equilibrium society for p=0.3, R=5 case is almost completely integrated: for this case, the computed value of S is 0.03 ± 0.03! Even for R=3, a very modest increase in one's vision, the value of S for the equilibrium society is already down to 0.16 ± 0.04! This result, in and of itself, is important.

Recall that we have chosen to concentrate on the worst-case scenario of c=0.5 (equal numbers of two races trying to live in the same cityscape). And, even in this worst case scenario, stable, integrated communities are formed with a rather modest increase in vision (R=3 to 5) and for significant non-zero values of p (0.3 in this case). We conclude that in order to have stable, integrated societies, it is not necessary for the agents to have utopian attitudes (actively seeking more diverse neighborhoods): this is not allowed in the present model. Nor is it necessary for the agents to have a Gandhian, color-blind world-view, where one does not care at all about the typology of one's neighbors: this would correspond to p=0. All one needs is a rather modest decrease in one's obsession with insisting that one must never be a minority in one's own neighborhood at any length scale (which is what p=0.5 means)! A decrease from p=0.5 to p=0.3 when combined with the powerful amplifying effect of even a modest increase in vision; from a myopic R=1 to a modest R=3 or 4, leads to stable, integrated societies!

The fact that stable, integrated neighborhoods form for such modest and eminently reasonable values of the parameters can have significant impact on the perspectives of policy makers. It provides some reason for the hope that reduction in racial neighborhood segregation—even complete integration—is a politically and socially viable goal. This result is also reassuring from another point of view. Recall that there is robust empirical demographic evidence (Farley and Frey 1991); (Farley et al. 1993) that there has been some significant decrease in the intensity of racial segregation in small and medium cities in the United States. Conservative commentators (Thernstrom and Thernstrom, 1997) have labored to make this point, but usually in the context of challenging what they believe to be exaggerated claims of liberal scholars or activists regarding the extent of racial neighborhood segregation. Our work suggests that we should not abandon Schelling type models: when extended to include agent-vision, they have the potential of giving us useful insights *and* of being consistent with empirical findings. Our work strongly supports the belief (Carr, 1999) that "Initiatives aimed at changing perceptions that fuel the desire to segregate will have a broader impact on reducing or eliminating segregation". Our simulations also lends some theoretical support to two specific policy initiatives (Yinger, 1995): to improve the availability and the flow of housing market information (increase R) and to encourage home-seekers to

consider alternate neighborhoods where their own race is not concentrated (increase R, effectively encourage a decrease in p).

We have discovered that the phase diagram of this model is much richer than previously believed: there are two distinct regimes of behavior in this model. In one regime, typified by p=0.5, initially integrated cityscapes segregate, the value of S increases with time, and it approaches a large value at equilibrium. This equilibrium segregation, $S(R)$, increases if R is increased: we call this the *unstable* regime. In the other regime, exemplified by p=0.3, initially integrated cityscapes segregate very little and S approaches a small value at equilibrium. This equilibrium segregation, $S(R)$, decreases if R is increased: we call this the *stable* regime. Fig. 5 graphically summarizes our results for $S(R)$, $1 \leq R \leq 7$ for the cases p=0.3 and p=0.5: the bifurcation and the two regimes are quite self-evident in this phase-diagram. To the best of our knowledge, this is a new technical result, whose importance lies in suggesting a new way of talking about the relation between agent-intent, agent-vision and the degree and nature of segregation in this and related models.

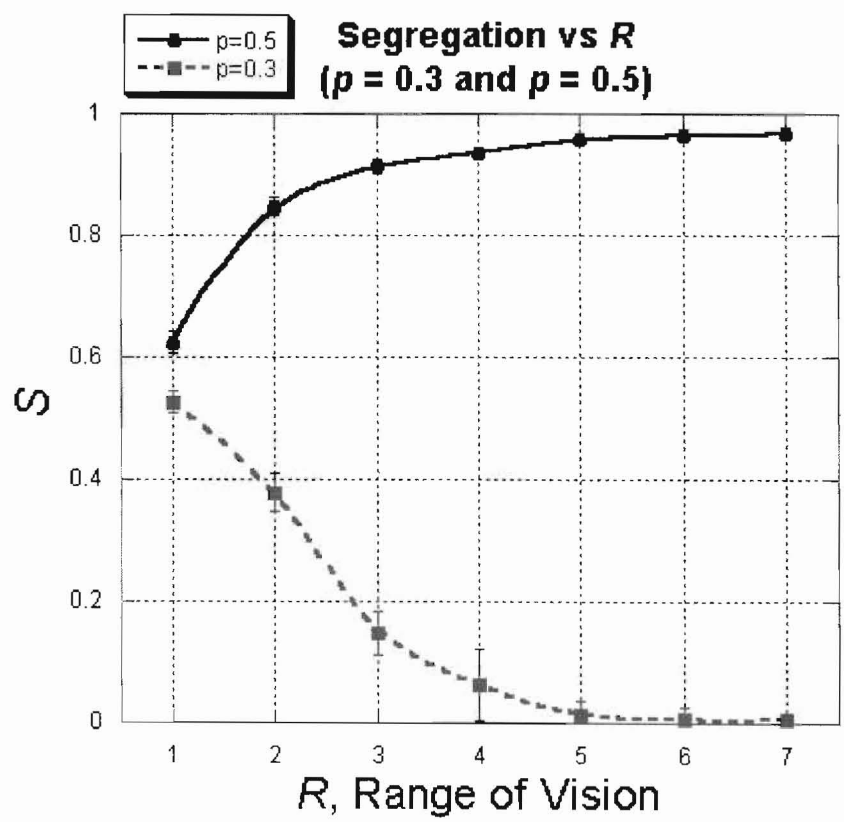

Fig. 5
Segregation vs. *R* in the *Unstable* and *Stable* Regimes

The behavior of the simulations in the *stable* regime is actually a bit more straightforward than the behavior in the *unstable* regime. For a moment, think about the case at hand, i.e. c=0.5, p=0.3. In an initial random society corresponding to these parameters, an agent has approximately 75% chance of having at least 30% of its von Neumann neighbors being of the same race. So, if R=1, about a quarter of the agents are dissatisfied, and move to 'better' locations. Since movement in this model always leaves behind some pockets of above-average concentration of minority agents, the chain-reaction does indeed lead to a modest amount of segregation for the R=1 case. But when the agents' vision is allowed to expand to 5, each agent has a very high probability (>99%) of seeing at least 30% of their own race. So, in the $R = 5$ simulations, very few agents are initially unsatisfied, and their moves cause very few, if any, of their far-sighted and fairly tolerant neighbors to become dissatisfied.

Thus, in the case where the agents don't insist upon quite as many of their neighbors being of the same race as the proportion occurring in the entire society, with high enough vision, segregation ceases to occur! In the extreme case, if the agents were to 'see' the entire society, movement would never occur, regardless of their preference for like neighbors because they would realize that their 'lot' could not be improved. Although most people, in reality, are likely to be more concerned with those in a fairly small neighborhood around them, even at $R = 5$, with p=0.3, we see virtually no segregation, and the society is already quite integrated at a modest value of R=3.

IV (c). Threshold (Critical) Value of Preference, p_c Separating the Two Regimes

The R=1 case of this model has a certain peculiar feature which, if viewed in isolation, can lead one to develop a false sense of a disconnection between agent-intent and the final equilibrium states of the cityscape: a case of 'the invisible hand' notions taken needlessly too far. As we indicated in the introduction, one example of such a sense of disconnect in the literature is provided by the claim, "...even quite color-blind preferences produced quite segregated neighborhoods" (Epstein and Axtell, 1996). In this section, we first indicate what this peculiar feature is and then go on to demonstrate that this peculiarity is erased for higher values of R, the range of vision. When the entire scene in the parameter-space is surveyed, there is no need to invoke any mysterious, invisible-hand, which somehow and unavoidably leads to segregation. A coherent and direct relationship between agent-intent and the degree and kind of segregation is restored.

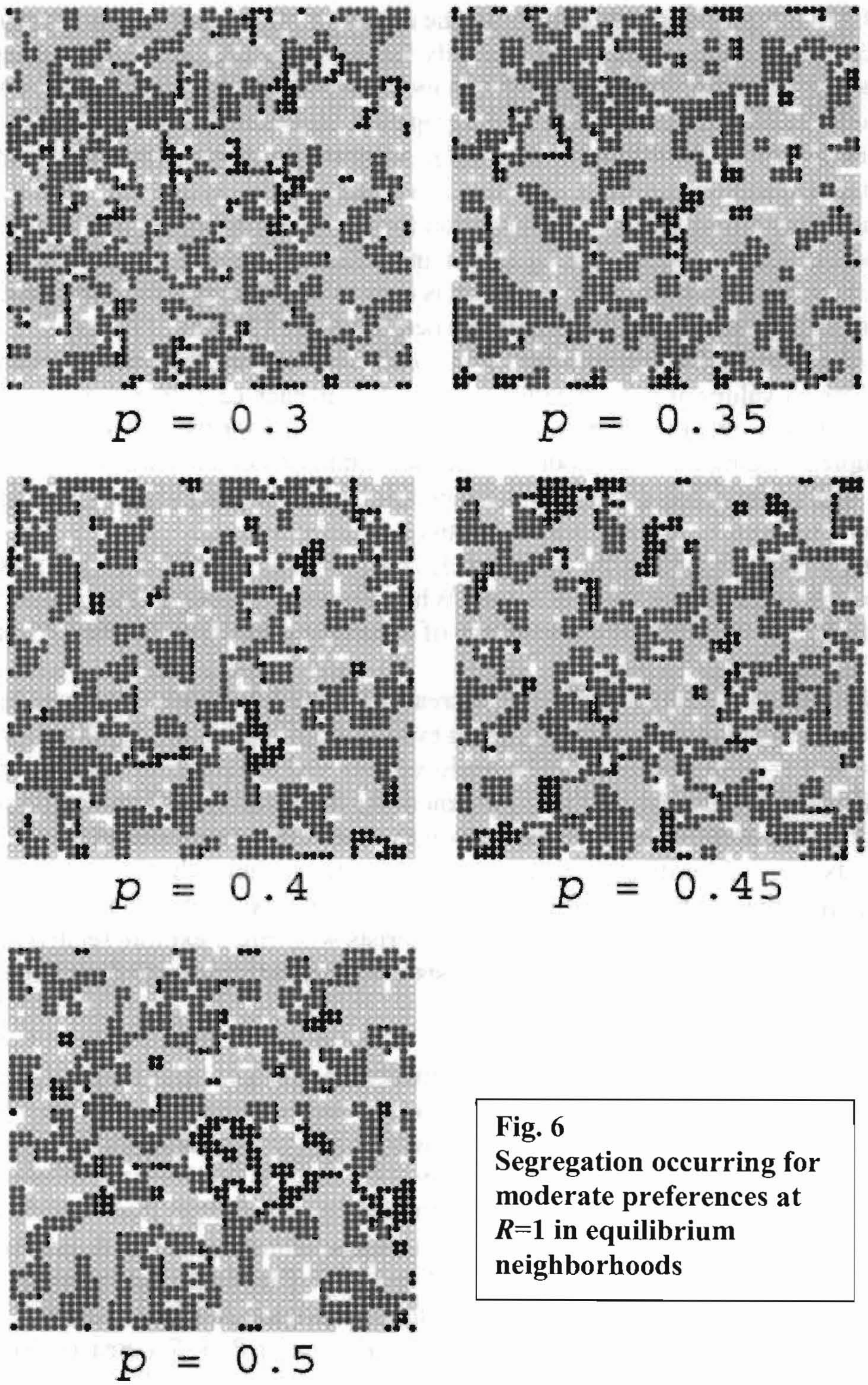

Fig. 6
Segregation occurring for moderate preferences at *R*=1 in equilibrium neighborhoods

The five panels in Fig. 6 display the equilibrium societies for a range of values of p from 0.3 to 0.5 in increments of 0.05 for the case R=1. If one had access to results only for this case of myopic vision (R=1), and only for this range of p, one could perhaps not be faulted too much for concluding that this model seems to lead, *almost always,* to a *certain* amount of segregation. And that segregation $S(p)$ seems to be very weakly dependent upon p, *in this range of values of* p, suggesting a disconnect of sorts between agent-intent (p) and outcome (S). But, this is not particularly mysterious. To understand this, think of the initial, random configuration for the case at hand, i.e. N=50, c=0.5, v=0.1 and calculate the number of agents having different numbers n of occupied neighbors (without regard to their color). It is easy to calculate that the approximate number of agents with 4,3,2 and 1 occupied neighbors is 1476, 657, 110 and 8 respectively. For the 1476 agents with n=4 occupied neighbors, their decisions are *identical* for all values of p in the range $0.3 \leq p \leq 0.5$: in each case, they require 2 neighbors of their own race to be *satisfied.* The same is true for the 118 agents with n=1 or 2 occupied neighbors: 1 neighbor of their own kind is needed to *satisfy* them. The only decision that varies in this range of p is the decision of 657 agents with n= 3 occupied neighbors who need 1 neighbor for $p \leq 1/3$ and 2 neighbors for $1/3 \leq p < 0.5$. So, one *expects* no difference among the cases p=0.35, 0.4, 0.45 and 0.5 and only a modest decrease in segregation for the p=0.3 case. When viewed in this light, the lack of variability in Fig. 6 is a trivial consequence of excessive granularity for the R=1 case.

Without the amplifying effect of increasing R, the bifurcation, that must happen somewhere between p=0.3 and p=0.5, (as is evident from Figs. 4 and 5) would not have been apparent at all to researchers who explored only von Neumann or only Moore neighborhoods. We use the symbol p_c to denote the threshold/critical value of p for which the initial slope of the $S(R)$ graph changes, from a negative value to a positive value). This threshold/critical preference, p_c demarcates societies that will tend to segregate from those where integrated neighborhoods will be stable. The specific value of this threshold p_c is likely to be of interest to theorists who might explore related models, to empirical demographers, and to sociologists who conduct surveys to determine agent preferences in contemporary communities.

Towards this end, it is useful to plot out $S(p,R)$ over a broader range of p, in small increments of p. There are two ways to display this three-dimensional data set. We present this data in Fig. 7 as a family of graphs $S(R)$ for various values of p. (For all values of $p \geq 0.5$, $S(R)$ increases monotonically. This range is therefore not displayed in the figure.) It is clear that the initial slope of the $S(R)$ graph changes from a positive value to a negative value at $p_c \cong 0.35$. For clarity, we also superimpose, in Fig. 8, graphs representing $S(p)$ for two values of R, i.e. R=1 and R=5. This figure demonstrates how increased vision strongly amplifies the dependence of S upon p. The very weak dependence for R=1 is transformed into a steep variation for R=5: S increases very rapidly between $0.32 < p < 0.36$ for R=5. Thus, for moderate and realistic values of R, a coherent and monotonic relationship between agent-intent (p) and outcome (S) is restored. The value of p for which S=0.5 may be taken as the dividing line, within this model, between segregated and integrated societies. This gives $p_c \cong$ 0.345. Thus, even

for the worst-case scenario of c=0.5 (the conflict of interest is maximized when equal numbers of two races try to live in the same cityscape), and for realistic values of R, approximately 35% of the parameter-space (p, R) leads to stable, integrated communities.

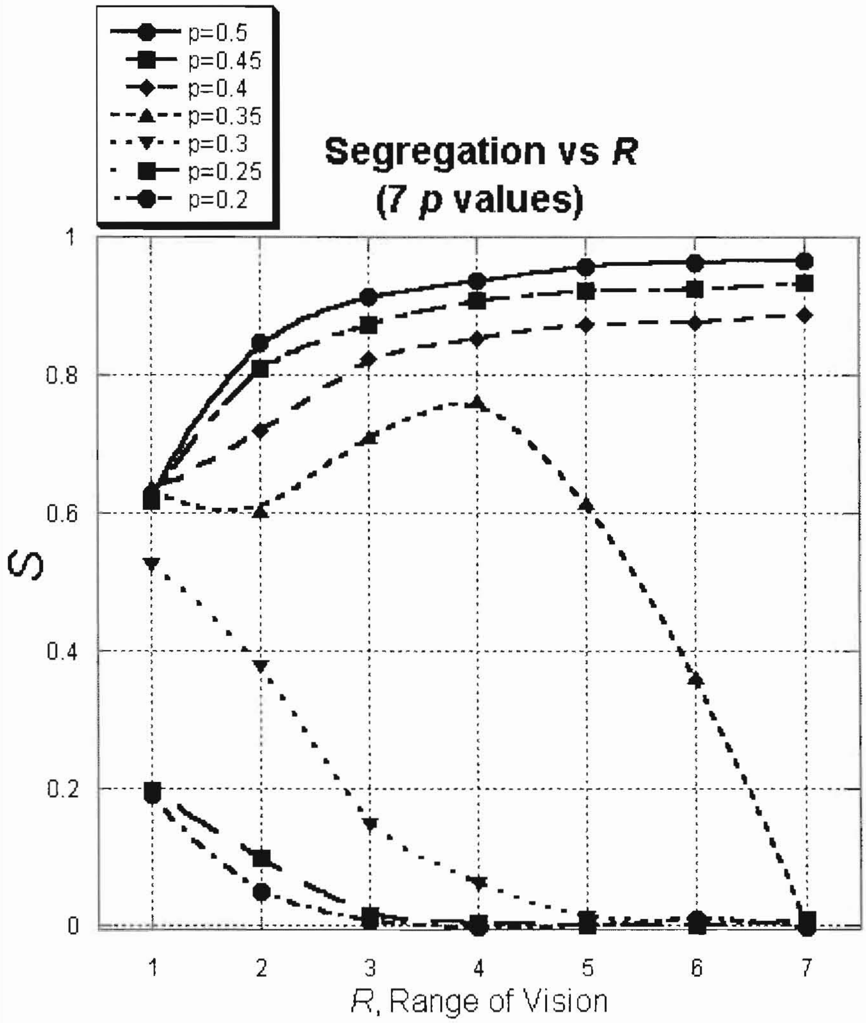

Fig. 7
Segregation vs. *R* for several different preferences

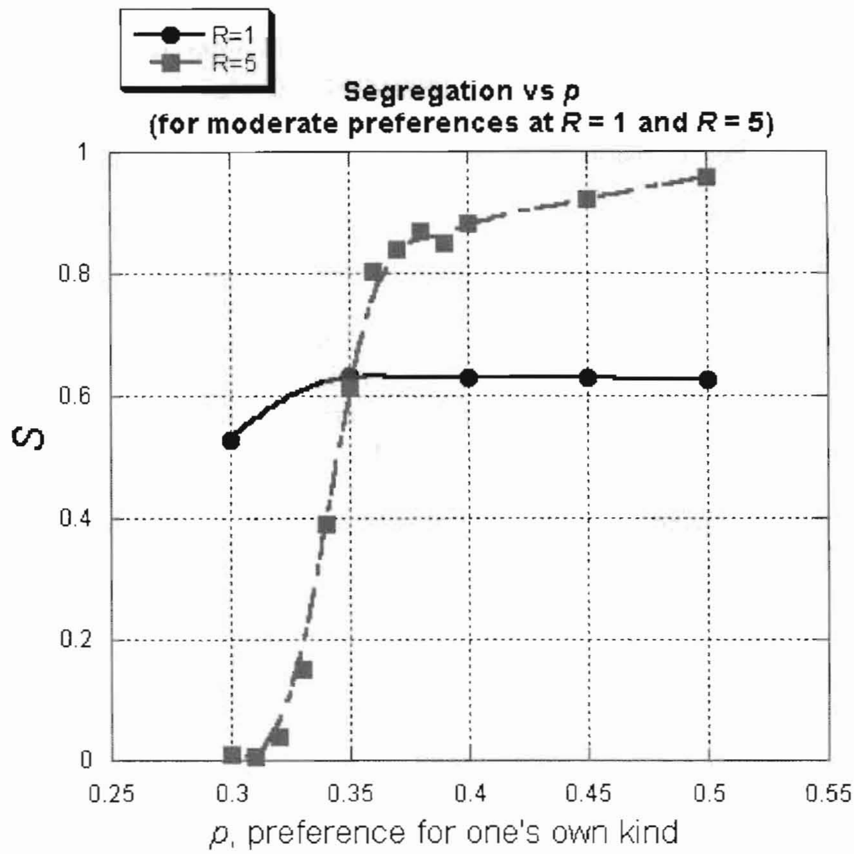

Fig. 8
Segregation vs. p at $R = 1$ and $R = 5$
Increasing R has the effect of amplifying the agent's preferences.

For the sake of completeness, we mention that there is a narrow, intermediate region of the parameter space ($p \cong 0.4$, $8 \leq R \leq 12$) where the system displays complex, metastable behavior, strongly resembling first order phase transitions in physical systems. We refrain from providing details because they are unlikely to have sociological significance.

VI. Conclusions

We have introduced and studied an extended Schelling model of racial neighborhood segregation, in which the agents authentically 'see' their neighbors up to a distance R; we call it the 'vision'. We have systematically and quantitatively explored the consequences of varying R and have developed a qualitative sense of how vision interacts with racial preferences p and minority concentrations c to lead to a non-simple segregation behavior.

We have discovered that the parameter space of this model has three regimes of behavior: the *unstable* regime, where the societies invariably segregate and segregation increases as vision, R, increases; the *stable* regime, where integrated societies are stable and segregation decreases as vision, R, increases; and a narrow *intermediate* regime where a complex behavior is observed.

The central policy implication of our study is an optimistic note: contrary to popular belief, rather modest decreases in xenophobia and/or preferences for one's own kind, *when coupled with increased vision*, can lead to stable and integrated neighborhoods. Public policy or procedures can *effectively* increase vision, e.g. realtors and clients could be provided with demographic data for $c(R)$ around various locations and/or tax incentives could be offered to avoid regions where fluctuations in $c(R)$ are above the global average. The education community and other social agents who work to lower preference for one's own kind and to increase tolerance for the 'other', can take strong encouragement from this study.

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CHAPTER TWO:

Suggestions of Critical Behavior Originating in the Schelling Segregation Model

I. Introduction

Our original intent in proposing a variant of the Schelling segregation model was to use knowledge from computational physics to inform upon sociological collective behavior. Specifically, the decision to incorporate vision was motivated by an analogy with Ising models. The phase diagram of Ising models with random exchange interactions, J_{ij}, (models which have some similarity to Schelling models with initially random racial configurations) is known to depend upon the range of the interactions. It seemed reasonable, therefore, for us to expect our variant of the Schelling model to exhibit different dynamics when different ranges of vision were employed.

Much to our delight, this interdisciplinary venture has proved to be doubly enlightening, as our work in sociology has delivered us a potentially important physical result. For simulations in our *intermediate* regime, we have observed dramatically large run-to-run fluctuations in equilibrium values of S. These run-to-run fluctuations are characterized by metastability and an extremely bimodal distribution of S, consisting of two very narrow peaks near $S\approx0$ (almost completely integrated) and $S\approx0.9$ (almost completely segregated) respectively: the open symbols in Fig. 2-1 show these peak values. The similarity to a first order transition from a 'supercooled' liquid-like to a 'superheated' solid-like phase is striking.

In order to study the possibility of a phase transition, in the traditional sense of the term, occurring in our model, we need to be able to control the governing parameters continuously. This is so because traditionally a phase transition is described by the discontinuity of a thermodynamic state function, or of one of its derivatives, as a field parameter is varied continuously. Typical examples include melting as a function of temperature and superconductivity as a function of temperature or magnetic field intensity. However, in our original model, R measures how far an agent sees from its own home as an integer number of houses. Since we can only assign discrete values to R, it is difficult to speak of a 'phase transition' as usually understood. We thus were motivated to propose a new model.

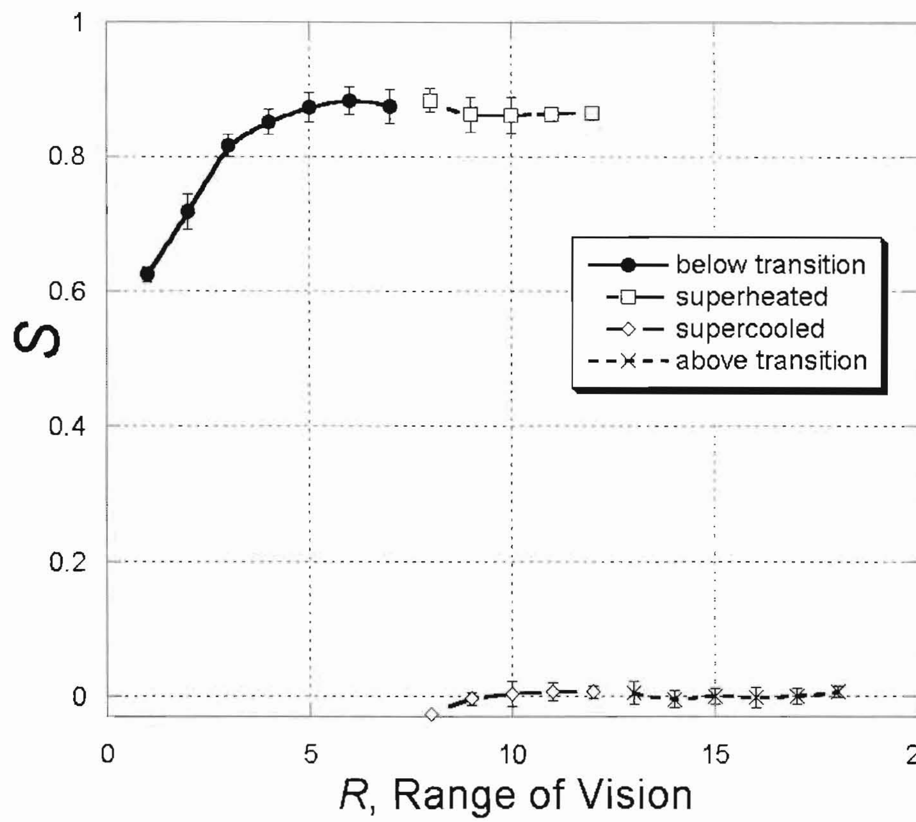


Fig. 2-1
***S* vs. *R* in the Intermediate Regime (*p* = 0.4)**

II. Our Continuous Model

We have designed a *continuous* model where the range of vision (denoted R_2 for notational clarity) can be varied continuously. In this model, racial preferences are modeled by a utility function which is maximized during simulation. Each pair of agents contribute a certain utility which depends upon the racial identities of the agents and upon their geographic distance, r_{ij}, from each other.

Our continuous model is a generalization of the model found in a study by Wasserman and Yohe (2001), which introduces a utility function that decreases exponentially with distance as a way to include the effects of the racial composition of neighbors further away from the agent's immediate neighborhood. Wasserman and Yohe's utility function incorporates an agent's desire to be near its own kind and the agent's desire to be far from the other kind, according to the formula:

$$U_j = \sum_{i=0}^{n} 2^{-(d(i)-1)} - \sum_{k=0}^{n} 2^{-(d(k)-1)}$$

where $d(i) \geq 1$ is the distance of a neighbor of individual j's own race, $d(k) \geq 1$ is the distance of a neighbor of a different race, and n is the number of neighbors within a range of vision. Notice that in Wasserman and Yohe's model, equal weight is attached to an agent's desire to be near its own kind and its desire to avoid the other kind. An agent will try to move if its utility falls below a certain value, and Wasserman and Yohe present results for simulations in which the threshold is zero. This is qualitatively similar to the moving criteria for our discrete model (for the case p=0.5, i.e. an agent will move if the (weighted or unweighted) fraction of like neighbors equivalently falls below 50%), except that in our discrete model the effects on utility do not decay with distance from the agent.

Our continuous model is an extension of our discrete model, but it also generalizes Wasserman and Yohe's model. The utility function which controls our continuous model is,

$$U_j = \frac{1}{R_2^{\,2}} \left(\sum_{i=1}^{l} e^{-\frac{r_i}{R_2}} - \mu \sum_{k=1}^{u} e^{-\frac{r_k}{R_2}} \right)$$

where r_i and r_k represent the distances[7] of the i^{th} and k^{th} agents, respectively, from the agent performing evaluation, l and u represent the number of like and unlike neighbors, respectively, μ is a parameter which indicates agent attitude, and R_2 represents a range of vision appropriate for our continuous model. While our utility function retains certain features of Wasserman and Yohe's model, such as exponentially-decaying utility and disutility contributions from like and unlike neighbors, we have added the two important parameters, μ and R_2, which bear qualitative similarity to p and R from our first model. Positive values of μ correspond to an agent wanting to be far from agents of the other race, but negative values of μ indicate that an agent wants to be near agents of the other race (possibly even more so than agents of its own race if $\mu < -1$). Note that Wasserman and Yohe's model is a special case of our continuous model – in the language of our continuous simulations, Wasserman and Yohe only use μ =1. The range of vision, R_2, controls how rapidly the magnitude of the utility contributions decay with distance from the evaluator. The factor of R_2^{-2} preceding the summations is merely a scaling factor to allow equal comparison of simulations with different R_2 values when a nonzero moving threshold is used.

[7] In this continuous model, the distance between agents is literally the closest distance from one site to the other, taking into consideration our periodic boundary conditions. In other words, $r = \sqrt{x^2 + y^2}$, where x is the fewest number of columns between agents counting either east or west and y is the fewest number of rows between agents counting either north or south.

Notice that in our discrete model, all occupied sites within an evaluating agent's range of vision contribute the same amount of utility (or disutility). Thus, for the discrete case $R = 7$, an agent's desires are equally impacted by those one home away and by those seven homes away. While this "square" utility function (see Fig. 2-2) may seem unrealistic, it was employed in our original work for the sake of simplicity. While the exponentially-decaying utility function (see Fig. 2-2) seems intuitively more realistic, we introduced it primarily because the continuity of R_2 allows us to study our model's possible critical behavior.

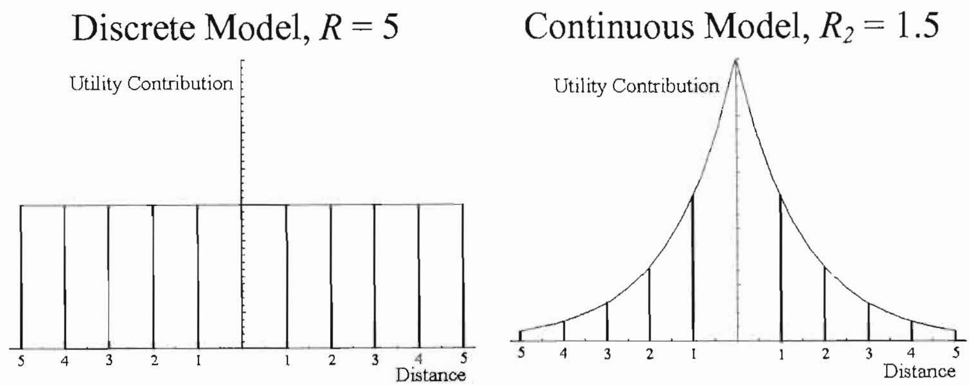


Fig. 2-2
Utility Contribution vs. Distance
The utility contributions of neighboring agents in our *continuous* model "decay" with distance from the evaluating agents; those in our *discrete* model do not.
Note: Agents can be non-integer distances away from the evaluator. For example, agents can be a distance of $\sqrt{2}$ or $\sqrt{5}$ from one another. The vertical lines seen in the figure, located at only integer distances, are meant to be only a guide to the eye.

III. Nature of the Phase Transition

In order to investigate the possibility of a phase transition in our model, we must determine whether several fundamental criteria are met. First, we must observe two distinct phases – one characterized by an order parameter near one and the other characterized by an order parameter near zero. By using our segregation metric, S, as an order parameter and by viewing the type of results displayed in Fig. 2-1, we certainly have fulfilled these initial two criteria. Second, we must determine if a phase *transformation* takes place, and if so, what type of phase transformation. Clearly, as the field variable, R, is increased, we see a transformation from the ordered phase ($S \approx 1$) to the disordered phase ($S \approx 0$) in the intermediate regime (see Fig. 2-1).

To answer the question of what type of phase transformation occurs requires a bit more subtlety. Fig. 2-1 was created using the results of 180 simulations. Using our discrete model with p=0.4, we performed 10 simulations each at R=1,2,…,18. The data points present in the ranges R=1 to R=7 and R=13 to R=18 represent the mean segregation of ten equilibrium ensembles. The 'error bars' in that range represent the standard deviation of segregation of these same ten ensembles. If we were to plot the results in the range R=8 to R=12 in the same manner, we would see essentially the same sort of behavior seen in the p=0.35 portion of Fig 7. The mean segregation value would appear neither near one nor zero, and the run-to-run standard deviation of segregation

would be quite large. Upon recognizing that the distribution of results in this region was extremely bimodal, we chose to plot the results near S=1 and those near S=0 separately.

By viewing the results of many simulations in which the same parameters are used, one can empirically construct a plot of the probability distribution of the order parameter, S. By repeating this process for several different values of the field variables (here, we use R and R_2 as field variables but we recognize that p and μ also can serve this purpose), we can discover how the probability distribution, $P(S)$, evolves as change in the field variables passes the system through the transformation point. In first order phase transitions, this distribution is known to exhibit bimodality at the transition point. This indicates a coexistence of the two phases at the transition point, exemplified by the coexistence of ice and water during melting. In higher order phase transitions, such as certain ferromagnetic transitions, bimodality is not present at the transition; a unimodal distribution merely shifts from the ordered regime to the disordered regime. An expository graphical representation of these different modes of transition can be found as Fig. 2-3.

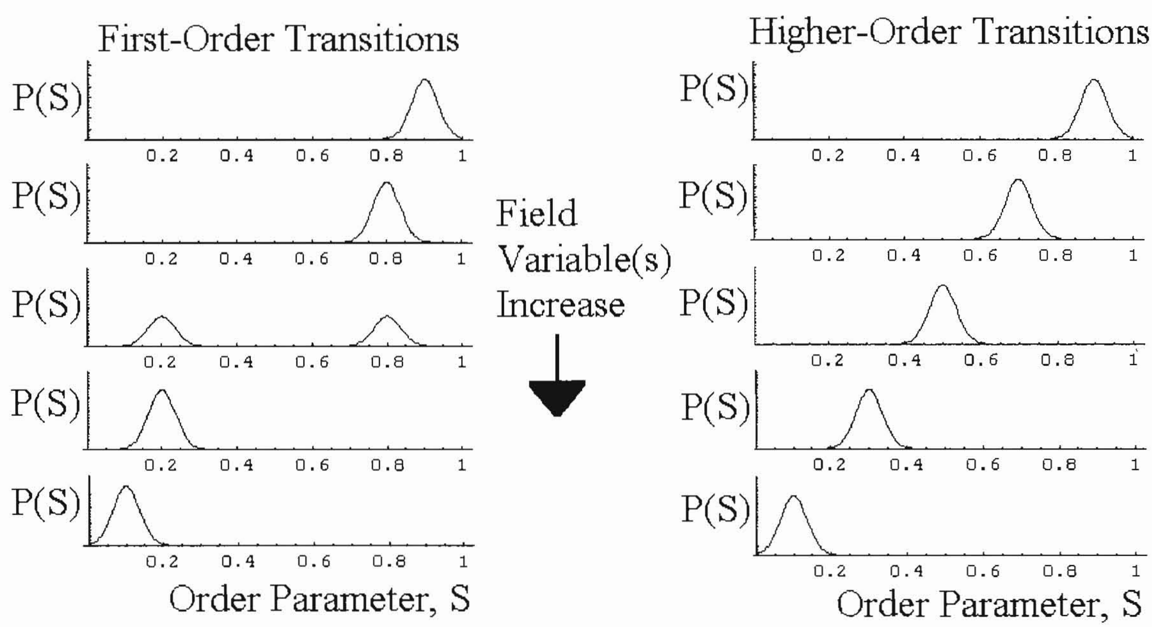


Fig. 2-3
Behavior of Order Parameter Distributions for First- and Higher-Order Transitions
In a first-order phase transition, the two phases coexist at the transition point. Thus, the order parameter exhibits a bimodal distribution as the field variable(s) pass through that point. In higher-order phase transitions, unimodality of this distribution is maintained throughout the transition.

IV. Results

All aspects of the behavior of our discrete model (an extension of Schelling's work) are replicated in our continuous model (a generalization of Wasserman and Yohe's work), including all three regimes of behavior, the persistent meta-stability, and occurrences of "superheating" and "supercooling". Thus, the shape of the utility contribution as a function of distance does not appear to have a fundamental impact upon simulation results.

A plot of segregation versus R_2 for our continuous model can be found as Fig. 2-4. By comparison with Fig. 7, it becomes clear that all three regimes of behavior are retained in our continuous model.

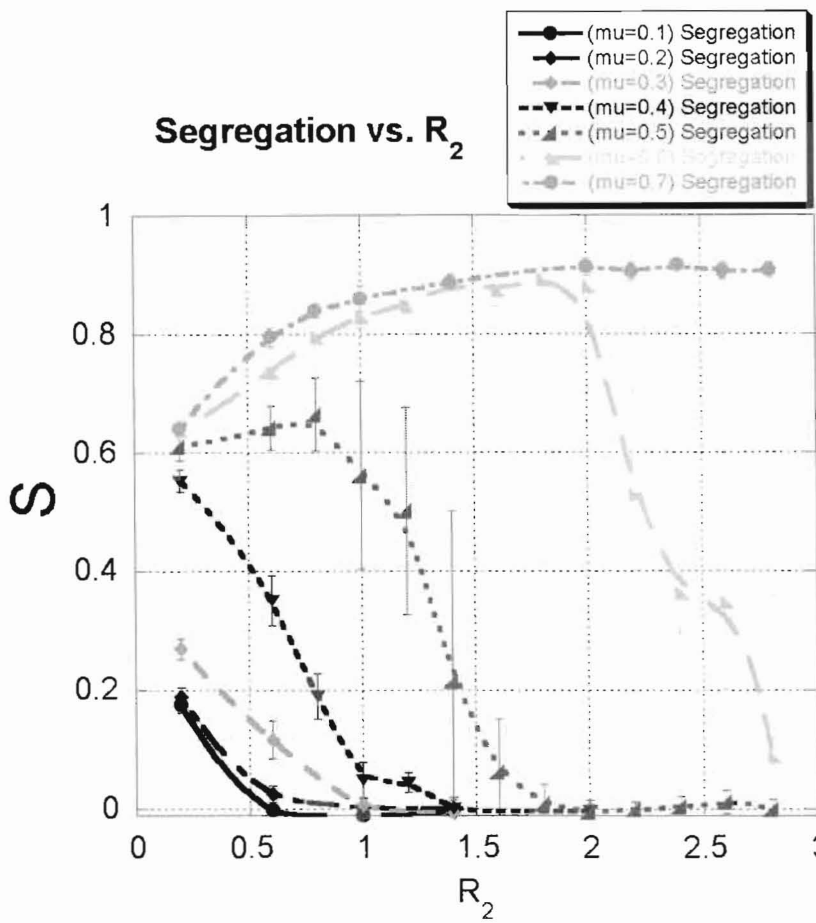


Fig. 2-4
Segregation vs. R_2 for Several Different Preferences
The continuous model exhibits the same qualitative behavior as the discrete model. All three regimes of behavior are retained.

Upon inspection of Fig. 2-4 it becomes clear that the behavior suggestive of a first-order phase transition is also present in our continuous model. In the range of parameter space (μ=0.6, 2≤R_2≤2.8), for example, we see run-to-run fluctuations indicative of the same sort of extreme bimodal distribution of segregation found in Fig. 2-1. This bimodal distribution resultant from our discrete model can be seen more explicitly as Fig. 2-5. Similarly, for our continuous model, the bimodal distribution suggested by Fig. 2-4 can be seen explicitly in Fig. 2-6.

In some regions of the parameter space, we observe a behavior more complex than that suggestive of a first-order phase transition. While the results presented in Fig. 2-6 (the μ=0.6, 2≤R_2≤2.8 region of parameter space) appear to be purely in the intermediate regime, results 'near the border' of the unstable and intermediate regimes can exhibit a different behavior. In the range (μ=0.5, 0.6≤R_2≤2.0), we see a distribution of results that broadens greatly as it passes through the transition point (see Fig. 2-7) yet does not appear to exhibit the bimodal distribution we discovered earlier.

Fig. 2-5 Distributions of the Order Parameter, *S*, as a Function of the Field Variable, *R*, in the *discrete* model

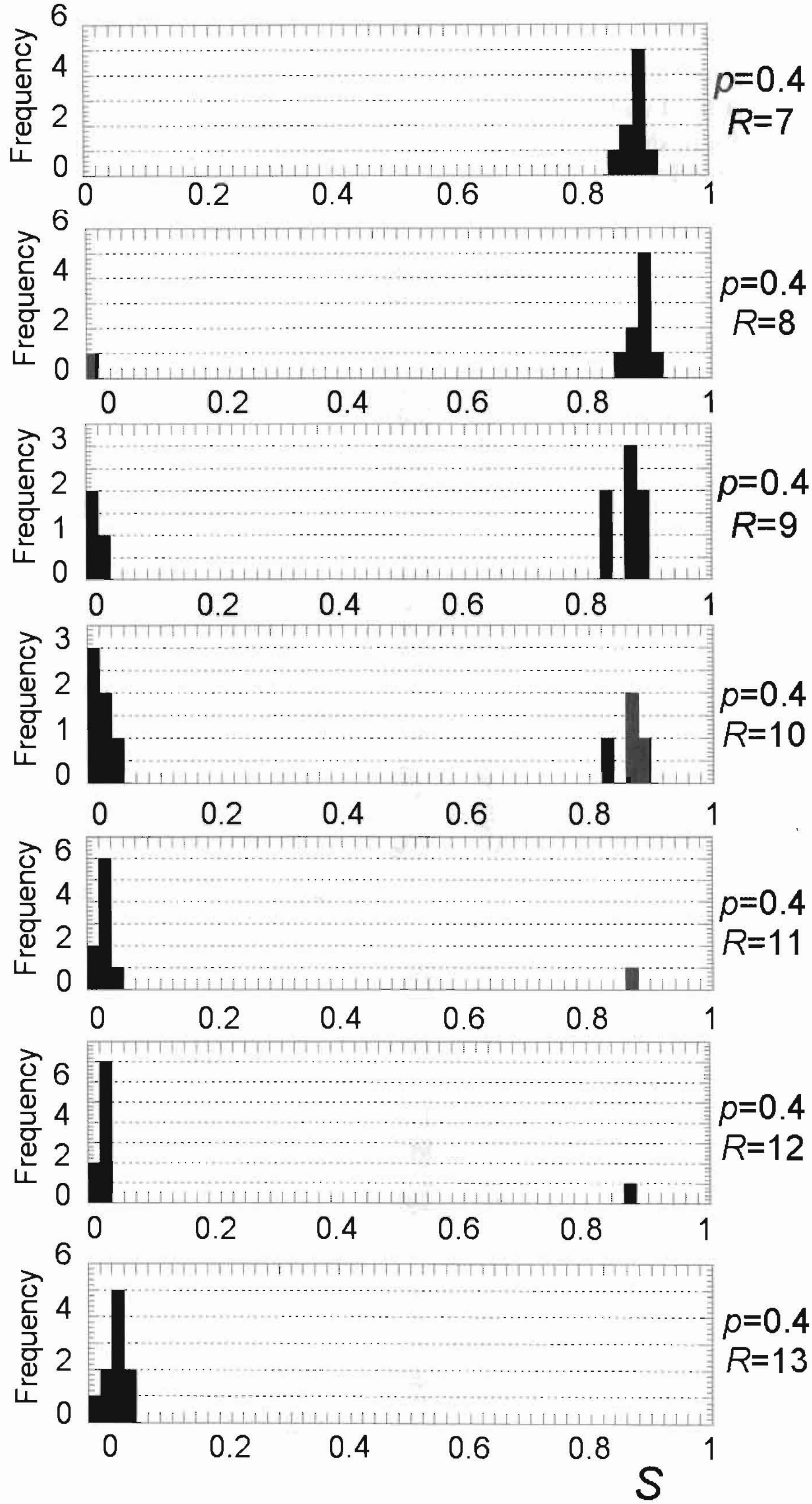


Fig. 2-6
Distributions of the Order Parameter, *S*, as a Function of the Field Variable, R_2, in the *continuous* model

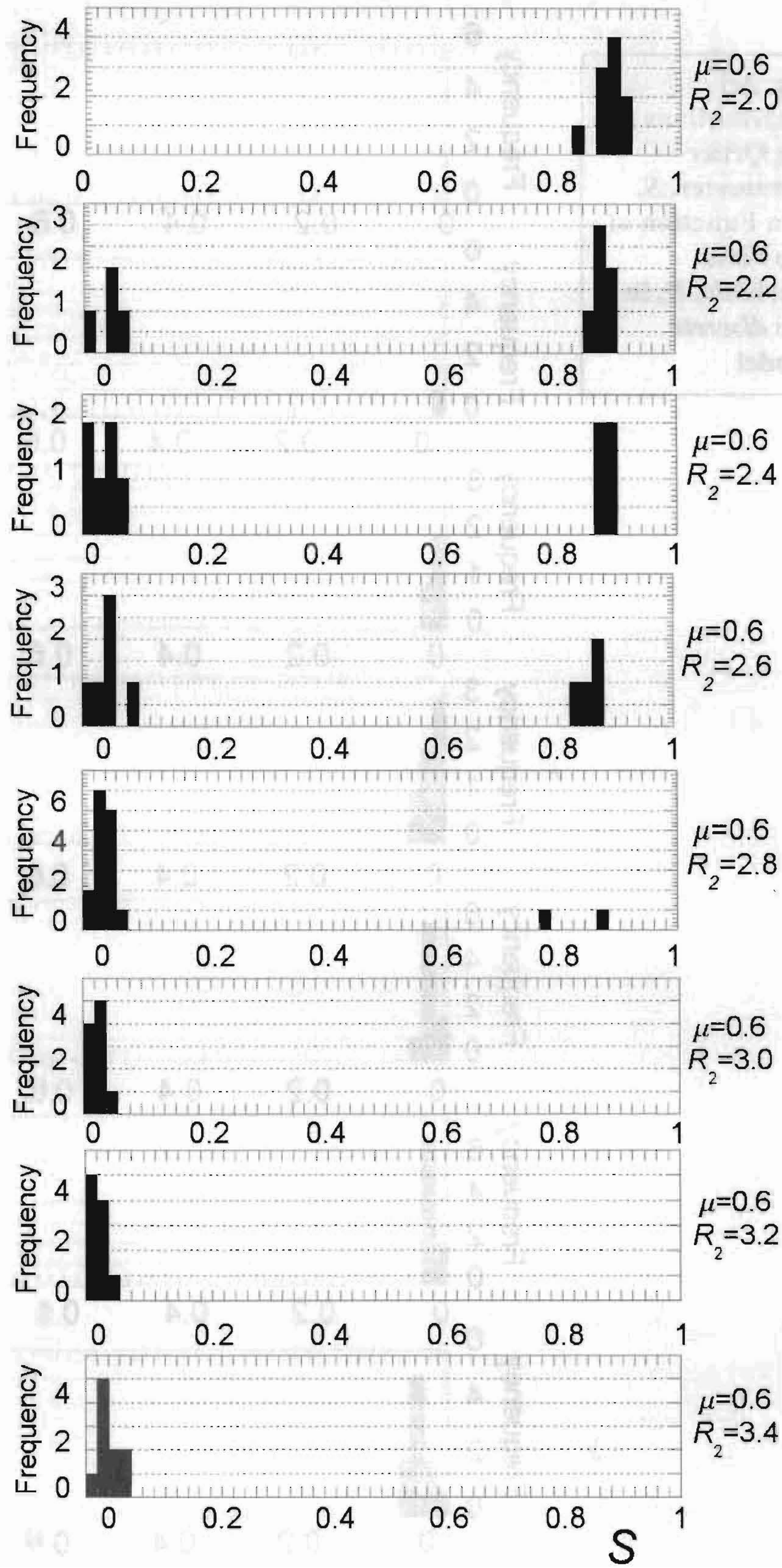
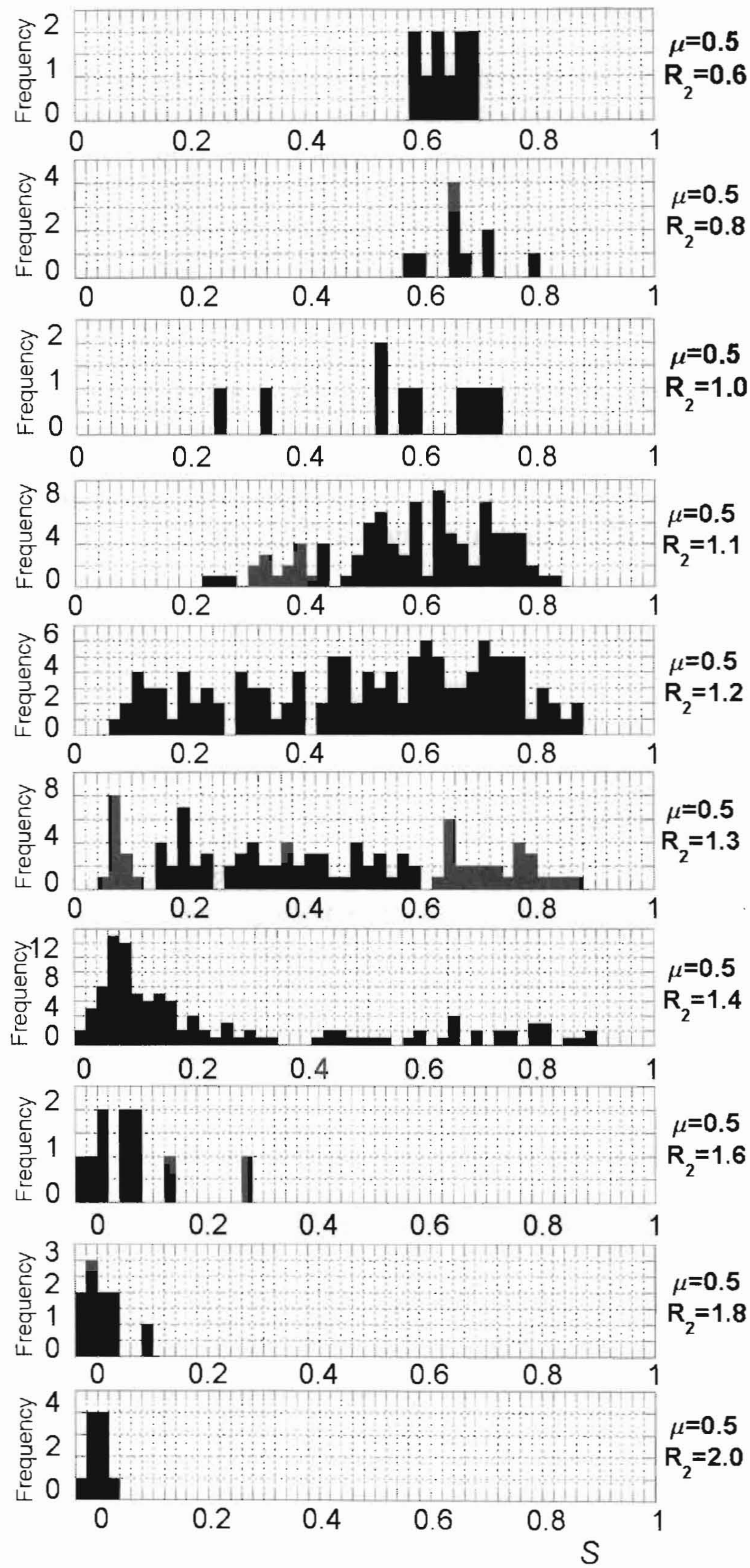


Fig. 2-7
Distributions of the Order Parameter, *S*, as a Function of the Field Variable, R_2, in our *continuous* model
In a portion of the parameter space where μ=0.5, we notice a behavior more complex than the bimodality observed in the intermediate regime.



V. Conclusion

The results in the intermediate regime of both of our models have similarities to first order phase transitions in physical systems and suggest lines of exploration that could be of significant interest to the physics community. The additional discovery of the new transition behavior witnessed in Fig. 2-7 hints that further investigation may be quite fruitful. We continue to study this phenomenon and its relationship to past similar findings to determine whether we are observing the appearance of previously discovered phenomena in a new setting or if we have, in fact, found compelling evidence for a new class of phase transitions.

The continuous model also demonstrates the robustness of all sociologically relevant conclusions drawn in chapter one. The discovery of the presence of the same three behavioral regimes and all associated trends in both our models confirms that our original results were robust and not merely algorithmic artifacts related to the specific treatment of vision used in our discrete model.

REFERENCE

WASSERMAN, H. and YOHE, G. (2001). Segregation and the provision of Spatially Defined Local Public Goods. The American Economist, Vol. 45: 13-24.