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Macroelements and Orthogonal Multiresolutional Analysis

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MACROELEMENTS AND ORTHOGONAL MULTIRESOLUTIONAL ANALYSIS.

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Orthogonal multiresolutional wavelet analysis in a two dimension setting furnishes a basis for wavelet analysis. Bernstein-Bezier polynomials over simplexes provide elegant expressions of the necessary and sufficient conditions for a shift invariant space generating an orthogonal multiresolution analysis. In order to give the expressions, a formula of the inner product of two Bernstein-Bezier polynomials over a simplex has been derived:

$$\langle P_n, Q_n\rangle_{\Delta s} = \iint_{\Delta s} P_n(x)\, Q_n(x)\, dx = S!V_{\Delta s} \sum_{i}\sum_{j} \mathbf{a}_i \mathbf{b}_j\, (n!)^2/(2n+s)! \prod_{k=1}^{s} \binom{i_k+j_k}{i_k}$$

where $V_{\Delta s}$ is the volume of the s-dimensional simplex, $\mathbf{i} = i_1+i_2+\dots+i_s$, $\mathbf{j} = j_1+j_2+\dots+j_s$, and $\mathbf{a}_i$ and $\mathbf{b}_j$ are respective Bernstein-Bezier coefficients of $P_n(x)$ and $Q_n(x)$. We also give the needed expression by using the formula above.

1. Introduction

1.1) Definition of $L^2(\mathbf{R})$ space.

$L^2(\mathbf{R})$ space is defined as a collection of square integrable functions, i.e. $\{f \mid \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\}$. $L^2(\mathbf{R})$ is also an inner product space in which an inner product, $\langle f,g\rangle$, $f,g \in L^2(\mathbf{R})$, is defined. Here the inner product can be understood as an extension of the dot product. The inner product is thus an infinite summation of dot products in essence. So,

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

whereas the dot product is denoted as:

$$\mathbf{v}\bullet\mathbf{g} = \sum_{i=0}^{n} v_i g_i.$$

for $\mathbf{v} = (v_0, v_1, \dots, v_n)$ and $\mathbf{g} = (g_0, g_1, \dots, g_n)$.

So, $L^2(\mathbf{R})$ space is defined as

$$L^2(\mathbf{R}) = \{f \mid \|f\|^2 = \langle f,f\rangle < \infty\}.$$

$L^2(\mathbf{R})$ is also a normed linear space, in which the norm $\|f\| = [\langle f,f\rangle]^{1/2}$ is defined.

1.15) Additional Definitions in $L^2(\mathbf{R})$.

It is helpful to the understanding of the following sections to define a few key concepts. The first necessary concept is orthogonality as understood in $L^2(\mathbf{R})$. Two functions are orthogonal if their inner product is equal to zero, i.e. $<f_1,f_2> = 0$. Orthonormal sets are sets of normalized orthogonal elements. Therefore an orthonormal set, which has an infinite number of elements, can be denoted as:

$$\{f_1,f_2,f_3,\ldots,f_n,\ldots\} \text{ such that } <f_i,f_j> = \delta_{ij}$$

where δ_{ij} is the delta function defined as $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i=j$.

The set $\{f_1,f_2,\ldots,f_n,\ldots\}$ in $L^2(\mathbf{R})$ is a complete orthogonal (or orthonormal) set if and only if $\forall f \in L^2(\mathbf{R})$, $<f,f_n> = 0$, $n = 1,2,\ldots$ implies $f = 0$ almost everywhere. Loosely understood every $f_i,f_j \in L^2(\mathbf{R})$ such that $<f_i,f_j> = 0$, where $f_i,f_j \neq 0$ must be a member of $\{f_1,f_2,\ldots,f_n,\ldots\}$ for the mentioned set to be complete. By almost everywhere we mean for all $x \in \mathbf{R}$ in the sense of some measurement $f(x) = 0$. If x has no measurement then $f(x)$ is not necessarily 0.

We define the term dense in $L^2(\mathbf{R})$ when refering to subspaces on $L^2(\mathbf{R})$. We say that a subspaces is dense in $L^2(\mathbf{R})$ if $L^2(\mathbf{R})$ is contained in the closure of the subspaces.

1.2) Approximation of $L(\mathbf{R})^2$ by use of orthonormal sets.

It can be shown that an orthonormal set, $\{f_1,f_2,\ldots,f_n,\ldots\}$, can be used to approximate functions in $L(\mathbf{R})^2$ if it is complete. First we begin with the assumption that a function $f \in L^2(\mathbf{R})$ can be expanded as:

$$f = \sum_{n=1}^{\infty} c_n f_n, \text{ where } \{f_1,f_2,\ldots,f_n,\ldots\} \text{ is orthonormal.}$$

Then $<f,f_n> = <\Sigma c_k f_k, f_n> = c_n.$

We must show that $f = \sum_{n=1}^{\infty} c_n f_n$ is true. To do so we must show that the sequence of partial sums of series converges, and that it converges to f. We first show that the $\sum_{n=1}^{\infty} c_n f_n$ is convergent.

Define $S_n = \sum_{k=1}^{n} c_k f_k$. If the limit of S_n exists as $n \to \infty$, then $\sum_{n=1}^{\infty} c_n f_n = \lim_{n\to\infty} S_n$. To show that $\lim S_n$ does exist, we consider Bessel's inequality, which states $\sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2$. In fact we consider the L^2 norm of f - S_n. (Note -- for a further understanding of Bessel's inequality, see reference)

(consider all Σ to run from 1 to ∞ unless otherwise noted in paper)

$$
\begin{aligned}
\| f - S_n \|^2 &= \langle f\text{-}S_n, f\text{-}S_n \rangle \\
&= \langle f - \Sigma c_j f_j, f - \Sigma c_k f_k \rangle \\
&= \int_{-\infty}^{\infty} (f - \Sigma c_j f_j)(f - \Sigma c_k f_k)dx \\
&= \int_{-\infty}^{\infty} (f^2 - f\Sigma c_k f_k - f\Sigma c_j f_j + \Sigma c_k f_k \Sigma c_j f_j)dx \\
&= \langle f,f \rangle - \langle f, \Sigma c_k f_k \rangle - \langle f, \Sigma c_j f_j \rangle + \langle \Sigma c_j f_j, \Sigma c_k f_k \rangle \\
&= \langle f,f \rangle - \Sigma C_k \langle f, f_k \rangle - \Sigma C_j \langle f, f_j \rangle + \Sigma\Sigma c_j c_k \langle f_j, f_k \rangle \\
&= \langle f,f \rangle - \Sigma |c_k|^2 - \Sigma |c_k|^2 + \Sigma |c_k|^2 \\
&= \|f\|^2 - \Sigma |c_k|^2.
\end{aligned}
$$

Since $\| f\text{-}S_n \|^2 \geq 0$, we obtain Bessel's inequality. The Bessel's inequality implies that $\Sigma |c_n|^2$ is convergent, because $\|f\|^2 < \infty$. Then the sequence of partial sums $S_n' := \sum_{k=1}^{n} |c_k|^2 > 0$ is increasing and bounded by $\|f\|^2$. It follows that $\{S_n'\}$ is a Cauchy sequence.

To show that the sum $\Sigma c_n f_n$ is convergent we will show that $\{\sum_{k=0}^{n} c_k f_k\}$ is a Cauchy sequence. We have shown that for every ε, there exists K such that for any n,m >K: $\| S_m\text{-}S_n \| < \varepsilon$. So,

$$
\begin{aligned}
\| S_m\text{-}S_n \| &= \left| \sum_{k=n+1}^{m} c_k f_k \right|^2 \\
&= \left\langle \sum_{j=n+1}^{m} c_j f_j, \sum_{j=n+1}^{m} c_j f_j \right\rangle
\end{aligned}
$$

$$= \langle \sum_{j=n+1}^{m} c_j f_j, \sum_{k=n+1}^{m} c_k f_k \rangle$$

$$= \sum_{j=n+1}^{m} c_j \left(\sum_{k=n+1}^{m} c_k \langle f_j, f_k \rangle \right)$$

$$= \sum_{j=n+1}^{m} |c_j|^2 < \varepsilon.$$

Therefore $\{S_n\}$ is a cauchy sequence, so $\Sigma c_n f_n$ converges.

To show that $\Sigma c_n f_n$ converges to f we must have the orthonormal set $\{f_1, f_2, \ldots, f_n, \ldots\}$ to be complete in $L^2(\mathbf{R})$ as previously defined. Because $\Sigma c_n f_n$ converges, let $g = \Sigma c_n f_n$, where $c_n = \langle f, f_n \rangle$ for each n. So,

$$\begin{aligned} \langle f-g, f_n \rangle &= \langle f - \Sigma c_k f_k, f_n \rangle \\ &= \langle f, f_n \rangle - \langle \Sigma c_k f_k, f_n \rangle \\ &= c_n - \Sigma c_n \langle f_k, f_n \rangle \\ &= c_n - c_n \\ &= 0. \end{aligned}$$

Since f-g = 0 almost everywhere, f=g. Therefore $\Sigma^{\infty} c_n f_n$ converges to f when $\{f_1, f_2, \ldots, f_n, \ldots\}$ is a complete orthonormal sequence.

1.3) Wavelets

A wavelet is defined as a single function with the property that the set of its dilations and translations, $\psi_{mn} = \{2^{m/2}\psi(2^m t - n)\}$, give a complete orthogonal (not necessarily but possibly orthonormal) basis of $L^2(\mathbf{R})$.

1.4) Examples of complete orthogonal bases of L^2

A) Trigonometric complete orthogonal system in $L^2(-\pi,\pi)$.

$L^2(-\pi,\pi)$ has a complete orthogonal basis $\{f_1, f_2, \ldots, f_n, \ldots\}$, where $f_1 = 1/2$, $f_2 = \sin x$, $f_3 = \cos x$, $f_4 = \sin(2x)$, $f_5 = \cos(2x)$, …. The inner product $\langle f_n, f_m \rangle$ is defined by $\int_{-\pi}^{\pi} f_n f_m dx = 0$ when $n \neq m$

and π when n=m. We can show $\{f_1,f_2,\ldots,f_n,\ldots\}$ to be complete on $L^2(-\pi,\pi)$. For any $f\in L^2(-\pi,\pi)$ we can find a sum $\Sigma c_n f_n = \Sigma<f,f_n>f_n$ that converges to f.

B) Wavelet Function

The Haar wavelet is denoted by $\varphi(t)$ and is defined as:

$$\varphi(t) = X_{[0,1]}(t),$$

where $X_{[0,1]}$ is the characteristic function defined as 1 on t = 0 to 1 and 0 elsewhere. The collection of its translations defined by $\varphi(t\text{-}n)$, $n\in N$, is an orthogonal system due to no overlap in translation; i.e. $<\varphi(t),\varphi(t-n)> = 0$ when $n\neq 0$. However, this system is not complete because $\varphi(t\text{-}n)$ is not dense in $L^2(\mathbf{R})$.

We use the function $\varphi_{mn}(t) := 2^{m/2}\varphi(2^m t-n)$. Then $\varphi_{mn}(t) = 1$ when $2^{-m}n \leq t < (n+1)2^{-m}$ and $\varphi_{mn} = 0$ elsewhere. Since any function in $L^2(\mathbf{R})$ can be approximated by piecewise constant functions that may have the jump at binary rational numbers, $\{\varphi_{mn}\}$is a complete system in $L^2(\mathbf{R})$. However $\{\varphi_{mn}\}$ is not an orthogonal set.

To construct a complete and orthogonal system in $L^2(\mathbf{R})$, we define a new function: $\psi = \varphi(2t) - \varphi(2t-1)$. The set $\{\psi_{mn} = 2^{m/2}\psi(2^m t-n)\}$ is a complete orthonormal system in $L^2(\mathbf{R})$. The system is complete because function in $L^2(\mathbf{R})$ can be expanded in $\varphi_{mn}(t)$. The system is orthonormal, which is confirmed by observation of the inner product $<\psi_{mn},\psi_{lk}>$ defined on the system.

$$
\begin{aligned}
<\psi_{mn},\psi_{lk}> &= \int_{-\infty}^{\infty} 2^{m/2}\psi(2^m t -n)2^{l/t}\psi(2^l t\text{-}k)\, dt \\
&= 2^{(m+l)/2}\int_{-\infty}^{\infty}\psi(2^m t\text{-}n)\psi(2^l t\text{-}k)\, dt \\
&= 2^{(m+l)/2}\int_{-\infty}^{\infty}\psi(t')\psi(2^{l-m}(t'+n)\text{-}k)\, 2^{-m}dt' \\
&= 2^{(m+l)/2}\int_{-\infty}^{\infty}\psi(t)\psi(2^{l-m}(t+n)\text{-}k)\, 2^{-m}dt \\
&= 2^{(m+l)/2}\int_{-\infty}^{\infty}\psi(t)\psi(2^{l-m}(t+n)\text{-}k)\, 2^{-m}dt
\end{aligned}
$$

Now when $t< 0$ or $t \geq 1$, $\psi(t)=0$, when $0\leq t< 1/2$, $\psi(t)=1$, and when $1/2\leq t< 1$ $\psi(t)=-1$. Thus,

$$2^{(l-m)/2}\int_{-\infty}^{\infty}\psi(t)\psi(2^{l-m}(t+n)-k)\,dt = 2^{(l-m)/2}\int_{0}^{1/2}\psi(2^{l-m}(t+n)-k)dt - 2^{(l-m)/2}\int_{1/2}^{1}\psi(2^{l-m}(t+n)-k)dt.$$

If $l\neq m$, the integrals are zero. When $l=m$, we have $\int_0^{1/2}\psi(t+n-k)dt - \int_{1/2}^{1}\psi(t+n-k)dt$. If $n\neq k$, the expression is equal to zero. If $n=k$, we have $\int_0^{1/2}\psi(t)dt - \int_{1/2}^{1}\psi(t)dt = 1$. Thus ψ_{mn} is orthonormal.

Since ψ_{mn} is a complete orthonormal basis of $L^2(\mathbf{R})$, for every $f\in L^2(\mathbf{R})$, we have the series $\bigoplus\bigoplus_{m,n=-\infty}^{\infty\ \infty} c_{mn}\psi_{mn} = \bigoplus\bigoplus_{m,n=-\infty}^{\infty\ \infty} <f,\psi_{mn}(t)>\psi_{mn}(t) = f$. If the approximate series begins from some m, then the series can be written as: $\bigoplus\bigoplus_{k,n=-\infty}^{m-1\ \infty} c_{kn}\psi_{kn} + \bigoplus_{k=m}^{\infty}\bigoplus_{n=-\infty}^{\infty} c_{kn}\psi_{kn}$. We define $W_m = \bigoplus_{n=-\infty}^{\infty} c_{mn}\psi_{mn}$ for a fixed m. We define $V_m = \bigoplus\bigoplus_{k,n=-\infty}^{m\ \infty} c_{kn}\psi_{kn}$. $V_{m+1} = W_{m+1} \oplus V_m$, where $\oplus$ denotes orthogonal union.

2. Multiresolutional Analysis

A multiresolutional analysis, denoted MRA, is a space denoted V_m, which is the span of $\{\phi_{mn}\}$. We define ϕ as a scaling function whose translates form a complete orthogonal basis of V_0 (span of $\{\phi_{0n}\}$, translates of ϕ with no dilation) . For multiresolutional analysis, we begin with a scaling function ϕ, which is a real valued function on $\mathbf{R}$ and generator of a multiresolutional analysis of $L^2(\mathbf{R})$ as follows:(Given by Walter [10])

(i) $\{\phi(t-n)\}$ is an orthonormal basis of V_0

(ii) $\ldots\subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbf{R})$

(iii) $f(t) \in V_m \Leftrightarrow f(2t) \in V_{m+1}$

(iv) $\bigcap_m V_m = \{0\}$, $\bigcup_m V_m = L^2(\mathbf{R})$

When there is a set of compactly supported functions which translate to form an orthogonal basis of V_0 we say that (V_m) is an orthogonal multiresolutional analysis(MRA). If the functions which form V_0 are continuous we say that (V_m) is a continuous multiresolutional analysis(MRA).

For univariate cases, when (V_m) is an orthogonal MRA it is possible to generate compactly supported wavelets that create an orthogonal basis of W_0. To achieve this, one must find the orthogonal scaling functions.

3) Orthogonal Finite Shift Invariant Spaces

Let Φ be a subset of $L^2(\mathbf{R})$ and $\tau(\Phi) = \{\phi(t-n) \mid n \in \mathbf{Z}, \phi \in \Phi\}$ give the set of integer translates of elements in Φ and let $\sigma(\Phi)$ be the L^2 closure of the linear span of $\tau(\Phi)$. Then $V \subset L^2(\mathbf{R})$ is a space called a finitely generated shift-invariant (denote FSI) space when $V = \sigma(\Phi)$ for a finite set Φ. If V_p is a MRA which is generated by r scaling functions, then V_p is a FSI space generated by $2^p r$ generators (refer to [3]).

4) Macroelements and their Bezier Coefficients

4.1) Definition of Bernstien Bezier coefficients.

We are able to define the functional value of a quadratic polynomial in in terms of Bernstien Bezier coefficients (denote Bezier Coefficient). We start by defining the baryocentric coordinate system. This coordinate system is defined over a triangle. We give the coordinate values of the three vertices $V_0 = (1,0,0)$, $V_1 = (0,1,0)$, and $V_2 = (0,0,1)$. Any point lying within the triangle can have its coordinates expressed in terms of this coordinate system. The triangle is subtriangulated into three subtriangles by connecting the chosen point with each vertex (see Figure 4.1.1). Then the areas of the subtriangles, a_0, a_1, and a_2 are found. The coordinates of the chosen point are then given by $(a_0/A, a_1/A, a_2/A)$, where A is the area of the triangle with vertices V_0, V_1, V_2. This gives us the baryocentric coordinate system. Thus we can express a point (x,y) in terms of baryocentric coordinates as follows

$$(x,y) = u_1V_0 + u_2V_1 + u_3V_2,$$

where $u_1 + u_2 + u_3 = 1$.

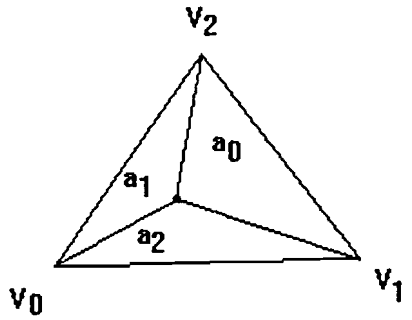

Figure 4.1.1

We can express a quadratic polynomial, P(x,y) in terms of it baryocentric coordinates by use of what are called Bezier coefficients. We express

$$P(x,y) = \sum \mathbf{a}_i (2!/\mathbf{i}!)\mathbf{u}^i$$

where $\mathbf{i} = (i_1,i_2,i_3)$, $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{i}! = i_1!i_2!i_3!$, $\mathbf{u}^i = u_1^{i1}u_2^{i2}u_3^{i3}$, and $\mathbf{a}_i$ are called Bezier coefficients with respect to P(x,y).

4.2) Definition of Macroelements

A macroelement is a piecewise function defined over a triangle under some subtriangulation (triangular partition). Macroelements can be used to interpolate the given function values and two partial derrivative values at the three vertices of the triangle.

4.3) Definition of C^1 continuity

For a function to have C^1 continuity, it must be differentiable all its first partialderivatives are continuous. Macroelements must be C^1 continuous over the interior and the edge.

4.4) Refinement of Macroelement

It is possible to use Bernstein-Bezier polynomials to generate smooth surfaces. Piecewise quadratic polynomials are the splines with the lowest possible degree and C^1 continuity. Macroelements are splines defined on a triangular region T. There are two main steps for construction of C^1 macroelements on T. First is to split the triangle into twelve subtriangular regions by using the three medians and line segments connecting the midpoints edges on T.

Next, to construct a quadratic element in each subtriangle under this division, use functional and first partial derivative values with respect to x and y at the three vertices of T. We denote this subtriangulation as the C^1 quadratic twelve-subtriangular macroelement.

Let the triangle $T = \langle V_1,V_2,V_3\rangle$, where $V_i = (a_i,b_i)$, $i = 1,2,3$. T is divided into twelve subtriangles as shown in Figure 4.4.1, where the medians V_1V_{23}, V_2V_{31}, and V_3V_{12} and the line segments $V_{12}V_{23}$, $V_{23}V_{31}$, and $V_{31}V_{12}$ are used in the subtriangulation. The medians are given by:

$V_{12} = 1/2(V_1 + V_2)$, $V_{23} = 1/2(V_2 + V_3)$, and $V_{31} = 1/2(V_3 + V_1)$.

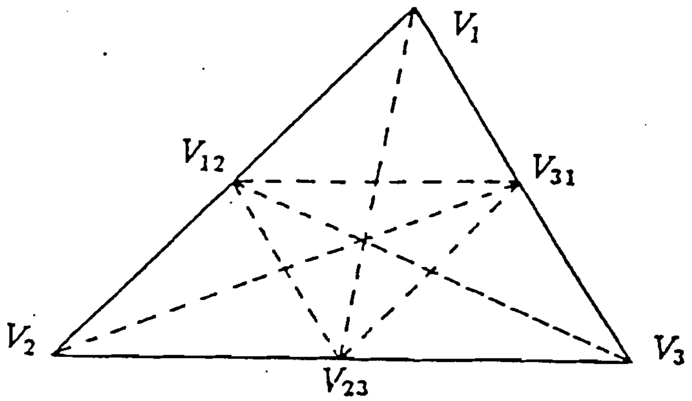

Figure 4.4.1

A quadratic polynomial defined on the triangle $\langle V_1,V_2,V_3\rangle$ can be expressed in terms of its Bezier coefficients $\mathbf{a}_i$, and the baryocentric coordinate system $(x,y) = u_1A + u_2B + u_3C$ as:

$P(x,y) = \sum \mathbf{a}_i\, (2!/\mathbf{i}!)\mathbf{u}^{\mathbf{i}}$, where $\mathbf{i} = (i_1,i_2,i_3)$, $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{i}! = i_1!i_2!i_3!$, and $\mathbf{u}^{\mathbf{i}} = u_1^{i1}u_2^{i2}u_3^{i3}$.

We will use the following parameters defined on T: d_i, m_i, n_i, and p_i. For $i = 1,2,3$ use d_i to denote the function value at V_i, m_i to denote the first partial derivative with respect to x at V_i, and n_i to denote the first partial derivative with respect to y at v_i. For I = 1,2,3 define p_i as follows:

$$p_1 = D_{k1}f(V_{12}),$$
$$p_2 = D_{k2}f(V_{23}),$$
$$p_3 = D_{k3}f(V_{31}),$$

with $\mathbf{k_1} = (b_1\text{-}b_2, a_2\text{-}a_1)$, $\mathbf{k_2} = (b_2\text{-}b_3, a_3\text{-}a_2)$, and $\mathbf{k_3} = (b_3\text{-}b_1, a_1\text{-}a_3)$. For any vector $\mathbf{k} = (k_1,k_2)$, D_kf denotes the derivative of f with respect to k given by

$$D_kf = \mathbf{k}\bullet\nabla f = k_1(\delta f/\delta x) + k_2(\delta f/\delta y).$$

Thus, p_1, p_2, and p_3 are normal derivatives at the midpoints of the edges V_1V_2, V_2V_3, and V_3V_1 of T, respectively.

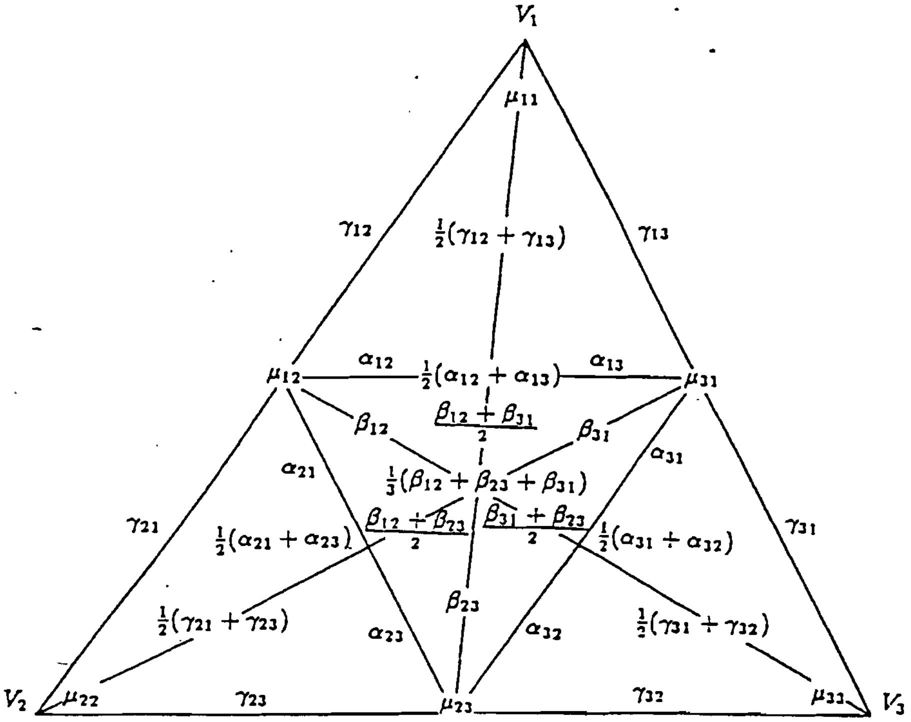

Figure 4.4.2

As shown in Figure 4.4.2, we can express all of the Bezier coefficients of the quadratic twelve subtriangulation macroelement on T. The respective Bernstien-Bezier coefficients shown in the subtriangulation are:(given by Chui and He [1]).

$$\mu_{ij} = (d_1 + d_j)/2 + 1/8[(m_i\text{-}m_j)(a_j\text{-}a_i) + (n_i\text{-}n_j)(b_j\text{-}b_i),$$

$$\gamma_{ij} = d_i + 1/4[m_i(a_j\text{-}a_i) + n_i(b_j\text{-}b_i)],$$

$$\alpha_{ij} = \mu_{ij} + 1/8[(a_k\text{-}a_j)\ \delta/\delta x\ f(V_{ij}) + (b_k\text{-}b_j)\ \delta/\delta y\ f(V_{ij})],$$

$$\beta_{ij} = \mu_{ij} + 1/24[(a_k - (a_i+a_j)/2)\ \delta/\delta x\ f(V_{ij}) + (b_k - (b_i+b_j)/2)\ \delta/\delta y\ f(V_{ij})]$$

Where i,j = 1,2,3, and k is the complement of {i,j} whenever $i \neq j$. The expressions necessary for finding the partial derivatives are:

$$(a_1-a_2)\ \delta/\delta x\ f(V_{12}) + (b_1-b_2)\ \delta/\delta y\ f(V_{12}) = 2D_{v1-v12}f(V_{12})$$
$$(b_1-b_2)\ \delta/\delta x\ f(V_{12}) - (a_1-a_2)\ \delta/\delta x\ f(V_{12}) = p_1$$

$$(a_2-a_3)\ \delta/\delta x\ f(V_{23}) + (b_2-b_3)\ \delta/\delta x\ f(V_{23}) = 2D_{v2-v23}f(V_{23})$$
$$(b_2-b_3)\ \delta/\delta x\ f(V_{23}) - (a_2-a_3)\ \delta/\delta x\ f(V_{23}) = p_2$$

$$(a_3-a_1)\ \delta/\delta x\ f(V_{31}) + (b_3-b_1)\ \delta/\delta x\ f(V_{31}) = 2D_{v3-v31}f(V_{31})$$
$$(b_3-b_1)\ \delta/\delta x\ f(V_{31}) - (a_3-a_1)\ \delta/\delta x\ f(V_{31}) = p_3.$$

5) Generating FSI over Simplexes

5.1) MRA in higher dimensions

For a given vector space in $\mathbf{R}^n$ with $e_1,\ldots,e_n$ independent vectors, then a sequence of nested closed linear subspaces (V_m) in $L^2(\mathbf{R})$ is a MRA of multiplicity r and dilation N if :

1) $f \in V_m \Leftrightarrow f(N^{-m}t) \in V_0$ for $N,m \in \mathbf{Z}$, $N > 1$

2) $\cap V_m = \{0\}$

3) $\cup V_m$ is dense in $L^2(\mathbf{R})$

4) There exists a set of r functions, $\phi_1,\ldots,\phi_r$, such that the set of lattice translates $\{\phi_s(t-m_1e_1-\ldots-m_ne_n) \mid s = 1,\ldots,r \text{ and } m_1,\ldots,m_n \in \mathbf{Z}\}$ form a complet orthogonal basis of V_0 (Given by Donovan, Geronomo, and Hardin [2]).

To satisfy these conditions, it is necessary to have the set of scaling functions ,$\phi_1,\ldots,\phi_r$, generate the MRA. If there is a set of scaling functions whose lattice translates create an orthogonal basis of V_0 we say that (V_m) is an orthogonal MRA.

To construct a set of scaling functions that satisfy the preceding properties, we have derived a useful formula:

$$<P_n, Q_n>_{\Delta S} = \iint_{\Delta S} P_n(x)\, Q_n(x)\, dx = S!\, V_{\Delta S} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j\, (n!)^2/(2n+s)! \prod_{k=1}^{S} \binom{ik+jk}{ik}, \quad S=2$$

where $V_{\Delta S}$ is the volume of the s-dimensional simplex, $\mathbf{i} = i_1+i_2+\ldots+i_S$, $\mathbf{j} = j_1+j_2+\ldots+j_S$, and $\mathbf{a}_i$ and $\mathbf{b}_j$ are respective Bernstein-Bezier coefficients of $P_n(x)$ and $Q_n(x)$.

5.2) Derivation of Inner Product Formula

We have two piecewise polynomials $P_n(x) = \sum \mathbf{a}_i \phi_i^n(\lambda)$ and $Q_n(x) = \sum \mathbf{b}_j \phi_j^n(\lambda)$ where $\phi_i^n(\lambda) = n!/\mathbf{i}!\ \lambda^{\mathbf{i}}$, $\phi_i^n(\lambda) = n!/\mathbf{i}!\ \lambda^{\mathbf{i}}$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mathbf{i} = i_1+i_2+i_3$, $\mathbf{j} = j_1+j_2+j_3$, and $\mathbf{a}_i$ and $\mathbf{b}_j$ are respective Bernstein-Bezier coefficients of $P_n(x)$ and $Q_n(x)$.

$$<P_n, Q_n> = \iint_{\Delta} P_n(x)\, Q_n(x)\, dxdy$$

$$= \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\, (n!)^2/(\mathbf{i}!\ \mathbf{j}!) \iint \lambda_1^{i1+j1} \lambda_2^{i2+j2} \lambda_3^{i3+j3}\, dxdy,$$

where $dxdy = \begin{vmatrix} \delta x/\delta\lambda_1 & \delta x/\delta\lambda_2 \\ \delta y/\delta\lambda_1 & \delta y/\delta\lambda_2 \end{vmatrix} \delta\lambda_1\, \delta\lambda_2 = \begin{vmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} \delta\lambda_1\, \delta\lambda_2 = V_\Delta\, \delta\lambda_1\, \delta\lambda_2$.

$$\sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\, (n!)^2/(\mathbf{i}!\ \mathbf{j}!) \iint_{\Delta} \lambda_1^{i1+j1} \lambda_2^{i2+j2} \lambda_3^{i3+j3}\, dxdy$$

$$= V_\Delta \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\, (n!)^2/(\mathbf{i}!\ \mathbf{j}!) \int_0^1 \int_0^{1-\lambda 1} \lambda_1^{i1+j1} \lambda_2^{i2+j2} (1-\lambda_1-\lambda_2)^{2n-(i1+j1)-(i2+j2)} d\lambda_2\, d\lambda_1.$$

Since,

$$\int_0^{1-\lambda 1} \lambda_2^{(i2+j2+1)-1} (1-\lambda_1-\lambda_2)^{[2n-(i1+j1)-(i2+j2)+1]-1}\, d\lambda_2 = (1-\lambda_1)^{[2n-(i1+j1)+2]-1}\, B(i_2+j_2+1,\ 2n-(i_1+j_1)-(i_2+j_2)+1),$$

we have

$$V_\Delta \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\, (n!)^2/(\mathbf{i}!\ \mathbf{j}!) \int_0^1 \int_0^{1-\lambda 1} \lambda_1^{i1+j1} \lambda_2^{i2+j2} (1-\lambda_1-\lambda_2)^{2n-(i1+j1)-(i2+j2)} d\lambda_2\, d\lambda_1$$

$$= 2V_\Delta \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\ B(i_2+j_2+1,\ 2n-(i_1+j_1)-(i_2+j_2)+1)[\,(n!)^2/(\mathbf{i}!\ \mathbf{j}!)]\ \int_0^1 \lambda_1^{i1+j1} (1-\lambda_1-\lambda_2)^{2n-(i1+j1)-(i2+j2)} d\lambda_1$$

$$= 2V_\Delta \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\ B(i_2+j_2+1,\ 2n-(i_1+j_1)-(i_2+j_2)+1)\ [(n!)^2/(\mathbf{i}!\ \mathbf{j}!)]\ B(i_1+j_1+1,\ 2n-(i_1+j_1)+2)$$

$$= 2V_\Delta \sum_i \sum_j \mathbf{a}_i \mathbf{b}_i\ [(n!)^2/(\mathbf{i}!\ \mathbf{j}!)][\,((i_2+j_2)!\ (2n-(i_1+j_1)+1)!)/(2n+2)!][\,((i_1+j_1)!(i_3+j_3)!)/(2n-(i_1+j_1)+1)!]$$

$$= \iint_{\Delta S} P_n(x)\, Q_n(x)\, dx = S!\, V_{\Delta S} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j\, (n!)^2/(2n+s)! \prod_{k=1}^{S} \binom{ik+jk}{ik},$$

where s=2.

5.3) Application of Inner Product Formula In generating MRA.

Using an example given by Donovan, Geronomo, and Hardin (refer to [3]), we can rework the example using our derived formula for inner products. Let $\phi_{k+1},\ldots,\phi_r$ be spanned by the function g_1 and $\phi_1,\ldots,\phi_k$ be spanned by g_0. We find that $\langle g_0,g_1\rangle \neq 0$, thus we must introduce a scaling function ω such that g_0 and g_1 are orthogonal i.e.

$$\langle g_0,g_1\rangle = \langle g_0,\omega\rangle^2/\langle\omega,\omega\rangle = 0$$

This above formula was given by Donovan, Geronomo, and Hardin as means of generating scaling functions necessany for a two dimensional MRA.

The formula for computation of $\langle P_n(x),Q_n(x)\rangle_{\Delta S}$ is utilized to provide the respective inner products.

Consider the equilateral lattice L in $\mathbf{R}^2$ generated by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (1/2,\sqrt{3}/2)$. We let Δ denote the triangle with vertices $(0, \mathbf{e}_1, \mathbf{e}_2)$ and Δ' denote the triangle with vertices $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2)$. We let r be the reflection across the common edge of the triangles. We allow T to be the triangulation that consists of Δ and Δ'. We define h to be the hat function of the lattice, that is h is a piecewise linear function on T and $h(i\mathbf{e}_1 + j\mathbf{e}_2) = \delta_{i,0}\,\delta_{j,0}$. An orthogonal MRA is generated from h.

We have that ω is expressed as a linear combination of $h\bullet u_{i,j}$, $\omega\bullet u_{i,j}$, and $\omega\bullet\mathbf{r}\bullet u_{ij}$ where $u_{ij}(x) = 3x - i\mathbf{e}_1 - j\mathbf{e}_2$ and $\tau_{i,j} = x - i\mathbf{e}_1 - j\mathbf{e}_2$ for all $x \in \mathbf{R}^2$.

We let $g_0 = hX_\Delta$, $g_1 = h\bullet\tau_{1,0}X_\Delta$, and $g_2 = h\bullet\tau_{o,1}X_\Delta$, where X_Δ denotes the characteristic function on Δ. We can compute the following inner products:

$$\langle h,1\rangle_\Delta = 2A\sum_{1,2,3}\sum \mathbf{a}_i\mathbf{b}_j\,(n!)^2/(2n+2)!\,\binom{i1+j1}{i1}\binom{i2+j2}{i2}\binom{i3+j3}{j3}$$

$$= 2(\sqrt{3}/2)\sum \mathbf{a}_i\,(1/4!)\binom{i1+1}{i1}\binom{i2+1}{i2}\binom{i3+1}{i3}$$

$$= 2(\sqrt{3}/2)(1/4!) \sum_{1,2,3} \mathbf{a}_i \binom{1}{1}^{+1}\binom{0}{0}^{+1}\binom{0}{0}^{+1}$$

$$= 2(\sqrt{3}/2)(1/4!)(1+0+0)(2)(1)(1)$$

$$= \sqrt{3}/12$$

and,

$$<h,h>_\Delta = 2A\sum_i\sum_j \mathbf{a}_i\mathbf{b}_j\,(n!)^2/(2n+2)!\,\binom{i1}{i1}^{+j1}\binom{i2}{i2}^{+j2}\binom{i3}{j3}^{+j3}$$

$$= 2(\sqrt{3}/4)(1/4!)\sum_{1,2,3}\mathbf{a}_i\,[1]$$

$$= (\sqrt{3}/2)(1/4!)1$$

$$= \sqrt{3}/24.$$

Hence, $<h,1> = \sqrt{3}/2$ and $<h,h> = \sqrt{3}/4$. Likewise $<g_0,g_1> = \sqrt{3}/48$ can be shown.

Because ω satisfies a nonhomogeneous dilation equation, it can be expressed in the form:

$$\omega = h\bullet u_{1,1} + \sum_{(i,j)\in Q} s_{i,j}\,\omega\bullet u_{i,j} + \sum_{(i,j)\in Q_r} s'_{i,j}\,\omega\bullet r\bullet u_{i,j}$$

where $Q = \{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0)\}$ and $Q_r = \{(0,0),(1,0),(0,1)\}$.

For ω to be continuous, $|S_{i,j}| < 1$ for $(i,j)\in Q$ and $|S'_{i,j}| < 1$ for $(i,j)\in Q_r$. Also because ω must be symmetric with respect to rotation leaving Δ fixed, we must have:

$$s_{0,0} = s_{2,0} = s_{0,2} = s_1$$
$$s_{0,1} = s_{1,0} = s_{1,1} = s_2$$
$$s'_{0,0} = s'_{1,0} = s'_{0,1} = s_3.$$

$$<\omega,1> = <h\bullet u_{1,1},1> + \sum_Q s_{i,j}<\omega\bullet u_{i,j},1> + \sum_{Qr} s'_{i,j}<\omega\bullet r\bullet u_{i,j}, 1>$$

$$= 1/9(<h,1> + \sum_Q s_{i,j}<\omega,1> + \sum_{Qr} s'_{i,j}<\omega,1>)$$

$$= (1/9)<h,1> + (1/3)\;(\sum_{i=1}^{3} s_i)<\omega,1>.$$

Solving for $<\omega,1>$ yields: $<\omega,1> = (1/3)<h,1>/(3-\sum_{i=1}^{3} s_i)$.

Since $X_\Delta = g_0 + g_1 + g_2$, we have: $<\omega,g_0> = (1/3)<\omega,1>$.

$<\omega,\omega>$ is calculated in a similar fashion:

$$\langle\omega,\omega\rangle = \langle h\bullet u_{1,1}, h\bullet u_{1,1}\rangle + 2\sum_{Q} s_{i,j}\langle\omega\bullet u_{i,j}, h\bullet u_{1,1}\rangle + 2\sum_{Qr} s'_{i,j}\langle\omega\bullet r\bullet u_{i,j}, h\bullet u_{1,1}\rangle$$

$$+ \sum_{Q}(s_{i,j})^2\langle\omega\bullet u_{i,j}, \omega\bullet u_{i,j}\rangle + \sum_{Qr}(s'_{i,j})^2\langle\omega\bullet r\bullet u_{i,j},\omega\bullet r\bullet u_{i,j}\rangle$$

$$= 1/9(\langle h,h\rangle + 2\sum s_{i,j}\langle\omega,h\bullet\tau_{1-i,1-j}\rangle + 2\sum s'_{i,j}\langle\omega\bullet r,h\bullet\tau_{1-i,1-j}\rangle + \sum(s_{i,j})^2\langle\omega,\omega\rangle + \sum(s'_{i,j})^2\langle\omega,\omega\rangle)$$

$$= (1/9)(\langle h,h\rangle + 6s_2\langle\omega,g_0\rangle + 6s_3\langle\omega,g_0\rangle + 3\sum_{i=1}^{3}(s_i)^2 \langle\omega,\omega\rangle).$$

$$\langle\omega,\omega\rangle = ((1/3)\langle h,h\rangle + 2\,(s_2 + s_3)\,\langle\omega,g_0\rangle)/(\,3-\sum_{i=1}^{3}(s_i)^2).$$

The equation from above, given by Donovan, Geronomo, and Hardin

$$\langle g_0,g_1\rangle = \langle g_0,\omega\rangle^2/\langle\omega,\omega\rangle$$

can be satisfied.

$$\langle g_0,\omega\rangle^2/\langle\omega,\omega\rangle = -33 + 54s_1^2 - 25s_1^2 + 18s_2 - 6s_1s_2 - 13s_2^2 + 18s_3 - 6s_1s_3 + 6s_2s_3 - 13s_3^2$$

$$= 0.$$

For ω to be continuous, $|s_i| < 1$. Values can be found where s_1, s_2, and s_3 fufill the above stated criteria. The values of the s_i, I=1,2,3 are used to find an ω such orthogonal scaling functions, ϕ^1, ϕ^2, ϕ^3, can then be found.

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For further discussion of baryocentric coordinates and Bezier coefficients, the reader is referred to [4, chapters 8, 18]

For further discussion of C^1 continuity, the reader is referred to [4, chap7.3]

For further discussion of Bessel's inequality, the reader is referred to [5, page 150] and [9, pages173 and 252]