The B-spline Wavelet Recurrence Relation and B-spline Wavelet Interpolation

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Contents

1. Introduction

In most signal processing applications, a given range of data is best described by a set of local characteristics as opposed to a single global characteristic. In image processing, for example, a region of an image that contains numerous edges is best described as a region whose pixel color values change abruptly,i.e., they are not continuous values of color. A region of constant color, or gradually changing color, is best described as a region whose pixel values are constant, or whose values increase linearly by some factor. It is advantageous to represent this data with signals capable of adapting to these types of local characteristics, as opposed to choosing the best global characteristic. Here, the B-spline wavelet recurrence relation is presented. The B-spline wavelet recurrence relation allows a wavelet of order $n + 1$ to be constructed from a wavelet of order n. This recurrence relation provides a mathematical tool capable of locally varying its degree of continuity. The order of differentiability of a B-spline wavelet increases as the order of the B-spline wavelet increases, and the values range from discontinuous to an arbitrary degree of continuity. A brief discussion of interpolation for splines and B-spline wavelets is introduced as a step toward a future application.

2. B-splines

B-splines have their roots in curve generation and computer-aided geometric design. The B-spline is a piecewise defined polynomial with a compact support. A B-spline of order m has degree $m - 1$.

B-splines are linear combinations of the basis function, $\phi_{ij}(u, v)$.

$$\phi_{ij}(u,v) = \frac{(m-1)!}{i!j!}u^i v^j$$

The basis function is a function of the barycentrical coordinates u and v where

$$u(x) = \frac{x_2 - x}{x_2 - x_1}$$

and,

$$v(x) = \frac{x - x_1}{x_2 - x_1}.$$

Note $u(x) + v(x) = 1$. Here, $x_2 > x_1$.

The following is the B-spline defined over the interval $[k, k+1]$:

$$N_m = \sum_{i+j=m-1} a_{ij}^m \phi_{ij}^{m-1}(u, v),$$

where a_{ij}^m are the Bezier coefficients.

The Bezier coefficients have the following recurrence relation:

$$a_{l+1}^n(k) = a_l^n(k) + \frac{1}{(n-1)}[a_l^{n-1}(k) - a_l^{n-1}(k-1)]$$

Here,

$a_{l+1}^n(k)$ is the $l+1$st Bezier coefficient of $N_n(x)$ over the interval $[k, k+1]$,
$a_l^n(k)$ is the lth Bezier coefficient of $N_n(x)$ over the interval $[k, k+1]$,
$a_l^{n-1}(k)$ is the lth Bezier coefficient of $N_{n-1}(x)$ over the interval $[k, k+1]$,
and $a_l^{n-1}(k-1)$ is the lth Bezier coefficient of $N_{n-1}(x)$ over the interval $[k-1, k]$.

The Bezier coefficient recurrence relation is from the following recursion formula for derivatives:

$$B_n'(x) = n\left[B_{n-1}(x) - B_{n-1}(x-1)\right].$$

3. B-spline wavelets

The B-spline wavelet is defined as follows:

$$\psi_m(x) = \sum_k q_{m,k} N_m(2x - k),$$

where

$$q_{mk} = \begin{cases} \frac{(-1)^k}{2^{m-1}} \sum_{l=0}^m \begin{pmatrix} m \\ l \end{pmatrix} N_{2m}(k+1-l) & 0 \le k \le 3m-2 \\ 0 & \text{otherwise} \end{cases}$$

and N_m is the mth order B-spline. Since this is just a linear combination of B-splines, the B-wavelet also has a compact support. From this mother wavelet, ψ_m, we can construct the following wavelet function that exhibits the translation and dilation of ψ_m:

$$\psi_{ij}(x) = 2^{i/2}\psi_m(2^i x - j).$$

The set of all ψ_{ij}, $\{\psi_{ij}\}$, forms the basis for the space of the square integrable functions,i.e., $L_2(R)$. Any function, f, in L_2 may be written as

$$f = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} c_{ij}\psi_{ij}.$$

We will now investigate how to determine the coefficients, c_{ij}. If $\{\psi_{ij}\}$ is orthogonal, i.e.,

$$\langle \psi_{ij}, \psi_{uv} \rangle = \delta_{iu}\delta_{jv},$$

we complete the inner product on both sides of the expression of f with ψ_{rs} and have

$$c_{rs} = \langle f, \psi_{rs} \rangle.$$

Thus, for any $f \in L_2$, we obtain

$$f = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \langle f, \psi_{ij} \rangle \psi_{ij}.$$

If $\{\psi_{ij}\}$ is not orthogonal, then we can construct a dual set, $\{\tilde{\psi}_{ij}\}$, such that $\{\psi_{ij}\}$ and $\{\tilde{\psi}_{ij}\}$ are biorthogonal, i.e.,

$$\langle \psi_{ij}, \tilde{\psi}_{uv} \rangle = \delta_{iu}\delta_{jv}.$$

As previously, we complete the inner product on both sides of the expression of f with $\tilde{\psi}_{rs}$. We have

$$c_{rs} = \langle f, \tilde{\psi}_{rs} \rangle.$$

Hence for any $f \in L_2$,

$$f = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \langle f, \tilde{\psi}_{ij} \rangle \psi_{ij}.$$

We have several special cases:

If $j=-1$, span of $\{2^{-\frac{1}{2}}\widetilde{\psi}(\frac{x}{2}-k)\}=W_{-1}$.

If $j=0$, span of $\{\widetilde{\psi}(x-k)\}=W_0$.

If $j=1$, span of $\{2^{\frac{1}{2}}\widetilde{\psi}(2x-k)\}=W_1$. So, L_2 is the direct sum, $\ldots,\oplus W_{-1}\oplus W_0\oplus W_1\oplus\ldots$, where the W_i's are mutually orthogonal.

4. B-spline wavelet recurrence relation

To formulate the recurrence relation, we first apply the Fourier transform to the mth order wavelet. We have,

$$\begin{aligned}\psi_m(x) &= \textstyle\sum_k q_{m,k}N_m(2t-k)\\ \widehat{\psi}_m(\omega) &= \textstyle\sum_k q_{m,k}N_m\widehat{(2t-k)}\\ &= \textstyle\sum_k q_{m,k}\frac{1}{2}e^{\frac{-i\omega k}{2}}\widehat{N}_m(\frac{\omega}{2})\end{aligned}$$

The following substitutions are performed consecutively: $z=e^{\frac{-i\omega}{2}}$ and $Q_m(z)=\frac{1}{2}\sum q_{m,k}z^k$.

$$\begin{aligned}\widehat{\psi}_m(\omega) &= \tfrac{1}{2}\textstyle\sum q_{m,k}z^k\widehat{N}_m(\frac{\omega}{2})\\ &= Q_m(z)\widehat{N}_m(\tfrac{\omega}{2})\end{aligned}$$

We have a similar result from applying the Fourier transform to the $(m+1)$th order wavelet.

$$\widehat{\psi}_{m+1}(\omega)=Q_{m+1}(z)\widehat{N}_{m+1}(\frac{\omega}{2})$$

Now, dividing one into the other,

$$\frac{\widehat{\psi_{m+1}(\omega)}}{\widehat{\psi_m(\omega)}}=\frac{Q_{m+1}(z)}{Q_m(z)}\frac{\widehat{N_{m+1}}(\frac{\omega}{2})}{\widehat{N_m}(\frac{\omega}{2})}.$$

Note that,

$$\widehat{N_m}(\frac{\omega}{2})=\left(\frac{1-e^{i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right)^m.$$

So, we have

$$\frac{\widehat{N_{m+1}}(\frac{\omega}{2})}{\widehat{N_m}(\frac{\omega}{2})}=\left(\frac{1-e^{i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right).$$

And thus substituting,

$$\frac{\widehat{\psi_{m+1}}(\omega)}{\widehat{\psi_m}(\omega)} = \frac{Q_{m+1}(z)}{Q_m(z)}\left(\frac{1-e^{i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right)$$

$$\widehat{\psi_{m+1}}(\omega) = \frac{Q_{m+1}(z)}{Q_m(z)}\widehat{\psi_m}(\omega)\left(\frac{1-e^{i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right).$$

Note, however, that

$$\widehat{N_1}(\frac{\omega}{2}) = \left(\frac{1-e^{i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right).$$

So,

$$\widehat{\psi_{m+1}}(\omega) = \frac{Q_{m+1}(z)}{Q_m(z)}\widehat{\psi_m}(\omega)\widehat{N_1}(\frac{\omega}{2}).$$

Now, define

$$\widehat{\bar{\psi_m}}(\omega) = \widehat{\psi_m}(\omega)\widehat{N_1}(\frac{\omega}{2}),$$

and note that

$$\bar{\psi_m}(x) = 2\psi_m(x) * N_1(2x),$$

where $*$ is the convolution operator. Further note that,

$$\begin{aligned} \bar{\psi_m}(x) &= 2\int_{-\infty}^{+\infty}\psi_m(x-u)N_1(2u)du \\ &= 2\int_0^{\frac{1}{2}}\psi_m(x-u)du. \end{aligned}$$

Regrouping, we have

$$\widehat{\psi_{m+1}}(\omega) = \frac{Q_{m+1}(z)}{Q_m(z)}\widehat{\bar{\psi_m}}(\omega).$$

Now, let

$$\frac{Q_{m+1}(z)}{Q_m(z)} = \frac{1}{2}\sum_k b_{m+1,k}e^{-i\frac{\omega}{2}k}.$$

So,

$$\begin{aligned} \widehat{\psi_{m+1}}(\omega) &= (\tfrac{1}{2}\textstyle\sum_k b_{m+1,k}e^{-i\frac{\omega}{2}k})\widehat{\bar{\psi_m}}(\omega) \\ \psi_{m+1}(x) &= \tfrac{1}{2}\textstyle\sum_k b_{m+1,k}\bar{\psi_m}(x-k) \\ &= \textstyle\sum_k b_{m+1,k}\int_0^{\frac{1}{2}}\psi_m(x-k-u)du. \end{aligned}$$

Letting $u = \frac{t+1}{4}$,

$$\begin{aligned}\psi_{m+1}(x) &= \textstyle\sum_k b_{m+1,k} \int_{-1}^{1} \psi_m(x - k - \frac{t+1}{4})\frac{1}{4}dt \\ &= \textstyle\frac{1}{4}\sum_k b_{m+1,k} \int_{-1}^{1} \psi_m(x - k - \frac{t+1}{4})dt.\end{aligned}$$

And, hence, we have arrived at the so-called B-spline wavelet recurrence relation.

5. Interpolation

A brief scheme for B-spline and B-spline wavelet interpolation is presented next. Interpolation has relevance in that it is a fundamental tool in curve generation, and every other application involving smooth, i.e., high-order, B-spline wavelets.

(a) B-spline interpolation First we consider B-spline interpolation.

A good function for interpolation must satisfy the conditions for a Lagrange interpolation:

$$If(x) = \sum_{j=\text{even}} f(j)g(x-j),$$

where $g(i-j) = \delta_{ij}$. We have another function called a quasi-interpolation:

$$Qf(x) = \sum_j \lambda f(\cdot - j)\phi(x-j),$$

where $\phi(x)$ is a linear combination of N_m, and λ, the linear functional, always satisfies the following conditions:

$$\begin{aligned}\lambda 1 &= 1, \\ \lambda x &= 0, \\ \lambda x^2 &= -\tfrac{1}{2}.\end{aligned}$$

The interpolations that will be constructed here will satisfy both sets of conditions, and will thus be the Lagrange interpolant with the highest possible approximation order.

We will develop a linear combination of $\phi(x)$, $\bar{\phi}(x)$, which is defined below. This $\bar{\phi}(x)$ is good because it is the most compact support for a quadratic basis for Lagrange interpolation.

$$\bar{\phi}(x) = \frac{3}{2}\phi(x) + \frac{1}{2}[\phi(x-1) + \phi(x+1)] - \frac{1}{4}[\phi(x-2) + \phi(x+2)]$$

We know that the Lagrange interpolant for the data given at even integers is: $\sum_{j=\text{even}} f(j)\bar{\phi}(x-j)$, where $\bar{\phi}(i-j) = \delta_{ij}$.

We need linear combinations of the following type:

$$\begin{aligned} \lambda 1f &= a_1[f(1) + f(-1)] + a_3[f(3) + f(-3)] \\ \lambda 2f &= a_0 f(0) + a_2[f(2) + f(-2)] + a_4[f(4) + f(-4)] \end{aligned}$$

So, we now have:

$$\sum_{j=\text{odd}} [\lambda 1f(\cdot + j)\phi(x-j)] + \sum_{j=\text{even}} [\lambda 2f(\cdot + j)\phi(x-j)] =$$

$$\sum_{j=\text{even}} f(j)\bar{\phi}(x-j).$$

We will use $\lambda 1f = \frac{1}{2}[f(j-1) + f(j+1)]$, and $\lambda 2f = \frac{3}{2}f(j) - \frac{1}{4}[f(j-2) + f(j+2)]$.

We then write these in the form of the Lagrange interpolant:

$$\sum_{j=\text{odd}} \{[\frac{1}{2}[f(j-1) + f(j+1)]]\phi(x-j)\} +$$

$$\sum_{j=\text{even}} \{[\frac{3}{2}f(j) - \frac{1}{2}[f(j-2) + f(j+2)]]\} =$$

$$\sum_{j=\text{even}} \{f(j)[\frac{1}{2}[\phi(x-j-1) + \phi(x-j+1)]\} +$$

$$\sum_{j=\text{even}} \{f(j)[\frac{3}{2}\phi(x-j-1) - \frac{1}{4}[\phi(x-j-2) + \phi(x-j+2)]]\} =$$

$$\sum_{j=\text{even}} \{f(j)[\frac{3}{2}[\phi(x-j)+\frac{1}{2}[\phi(x-j-1)+\phi(x-j+1)]-$$
$$\frac{1}{4}[\phi(x-j-2)+\phi(x-j+2)]]\} = \sum f(j)\bar{\phi}(x-j).$$

This $\bar{\phi}(x)$ is the interpolation spline.

Given the basis function, we now briefly consider B-spline wavelet interpolation.

(b) B-spline interpolation wavelet The wavelet interpolation function,$\bar{\psi}(x)$

is the wavelet function who's scaling function is the interpolation spline, i.e.,

$$\bar{\psi}(x) = \sum d_j \bar{\phi}(2x-j).$$

We now consider how to determine the coefficients d_j The following is the Fourier transformation of the spline interpolation wavelet :

$$\hat{\bar{\psi}}(\omega) = \frac{c(-z)}{c(z^2)} Q(z)\hat{\bar{\phi}}(\frac{\omega}{2}).$$

The d_j's are explicitly determined by

$$\sum d_j z^j = \frac{c(-z)}{c(z^2)} Q(z).$$

In this equation, we have:

$$c(z) = \frac{1}{4}z^{-1} + \frac{1}{4} + \frac{5}{4}z + \frac{5}{4}z^2 + \frac{1}{4}z^3 - \frac{1}{4}z^4,$$

and $Q(z) = \frac{1}{2}\sum q_k z^k$, with

$$q_{mk} = \begin{cases} \frac{(-1)^k}{4}\sum_{l=0}^{3} \begin{pmatrix} 3 \\ l \end{pmatrix} N_6(k+1-l) & 0 \le k \le 7 \\ 0 & \text{otherwise.} \end{cases}$$

6. Conclusion

As an aid to this research, two Mathematica packages were developed. The first package generates a B-spline of arbitrary degree. The second makes use of the first to generate a B-spline wavelet of arbitrary degree. In this paper, a recurrence relation for the B-spline wavelet was presented. It should be noted, however, that the existence of a recurrence relation is really just a convenience. In some cases, a recurrence relation makes certain algorithms or procedures practical from a computational point of view. The B-spline wavelet recurrence relation allows a higher order wavelet to be constructed from a lower order one, and it makes use of the computation already performed in computing the lower order wavelet. At this stage, the recurrence relation is just a theoretical tool that holds promise for future applications. The computation involved with the recurrence relation involves an expansion of coefficients and a numerical integration, both of which could be very expensive to perform. Future research might involve working out an interpolation example, theoretically, by making use of the B-spline wavelet recurrence relation. Theoretical schemes involving the recurrence relation could also be developed for image compression and curve editing .

7. Sample B-splines and B-spline wavelets

8. References

1. C.K. Chui, Introduction to Wavelets, Academic Press, 1991.
2. T.X. He, "Spline interpolation and its wavelet analysis", Approximation Theory VIII, Vol 2: Wavelets ans Multilevel Approximation, ed. C.K. Chui and L.L. Schumeker, World Scientific Publishing.
3. T.X. He, "Spline Wavelet Transforms", to appear.

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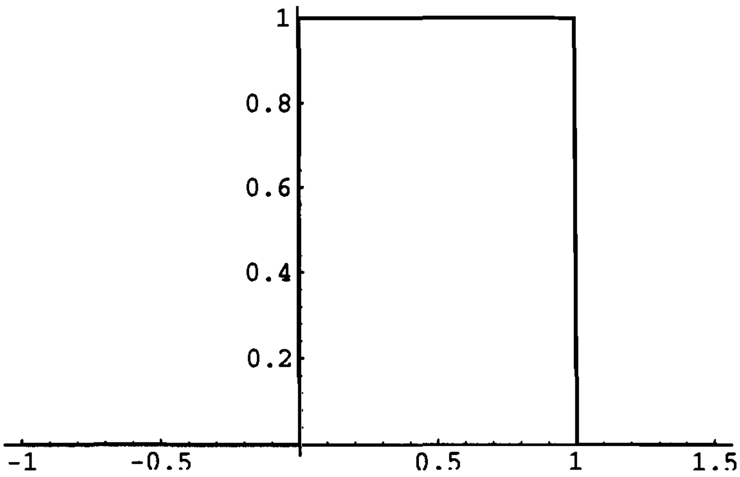

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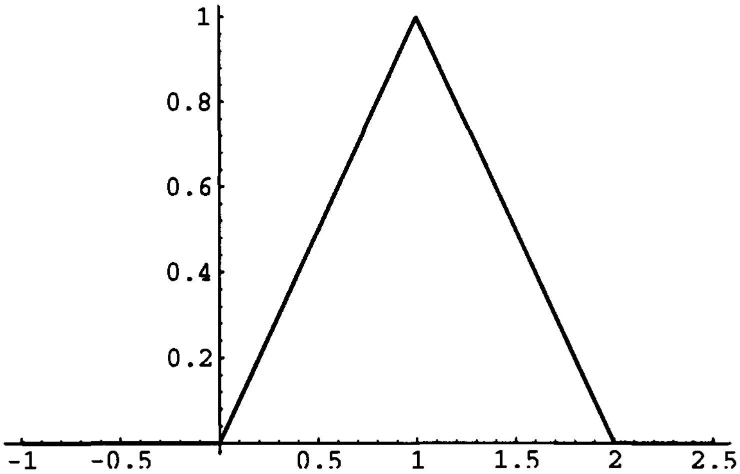

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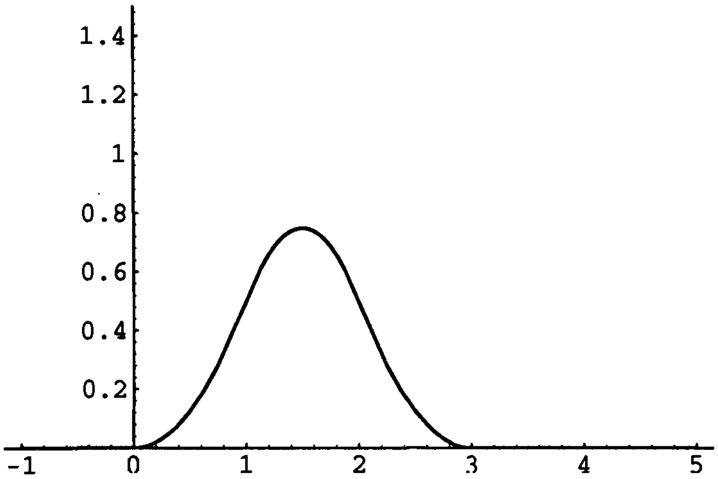

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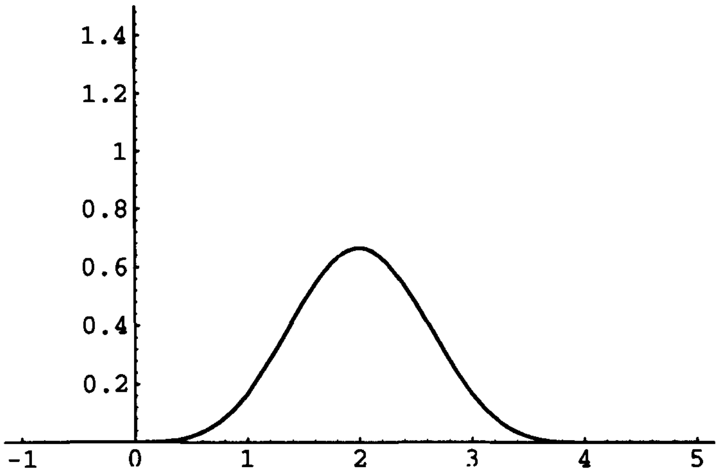

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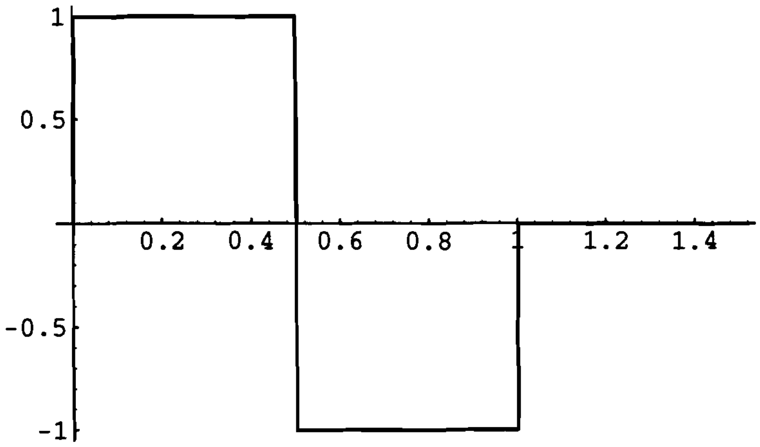

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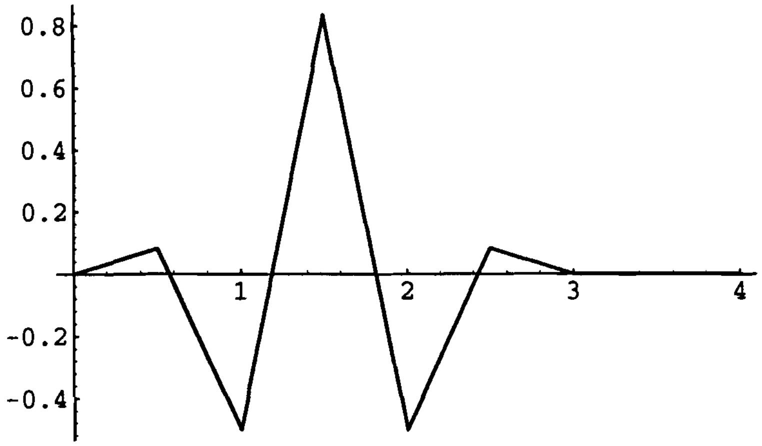

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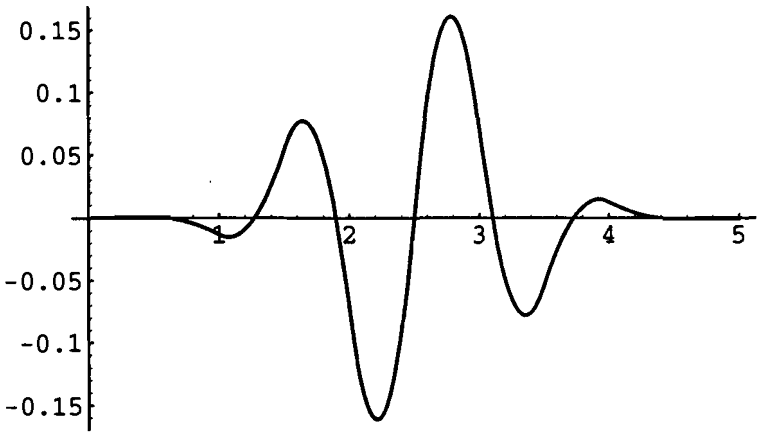

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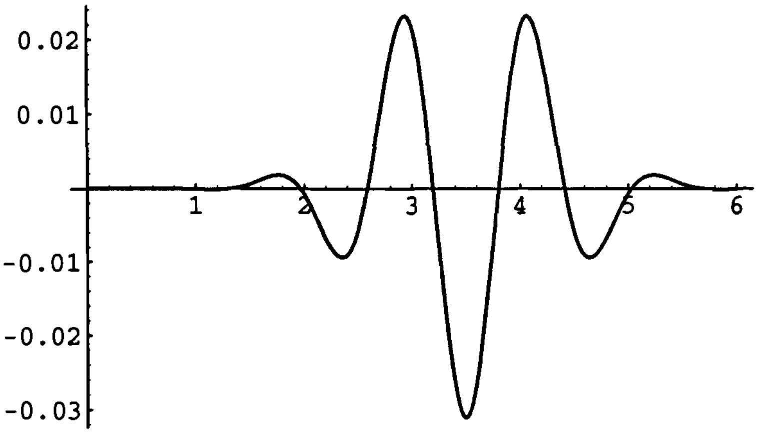

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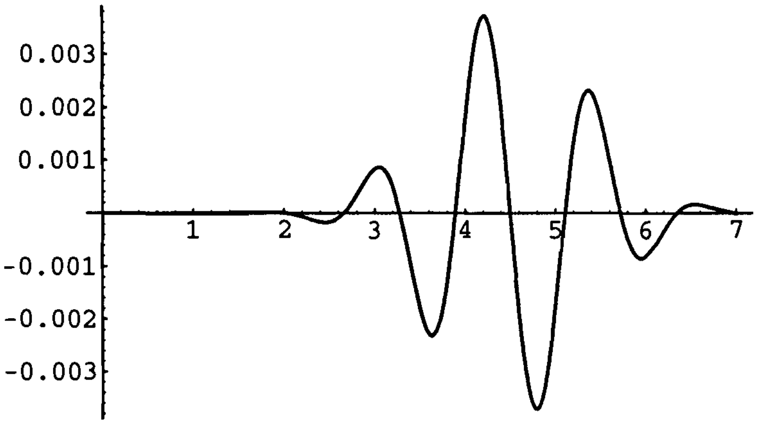

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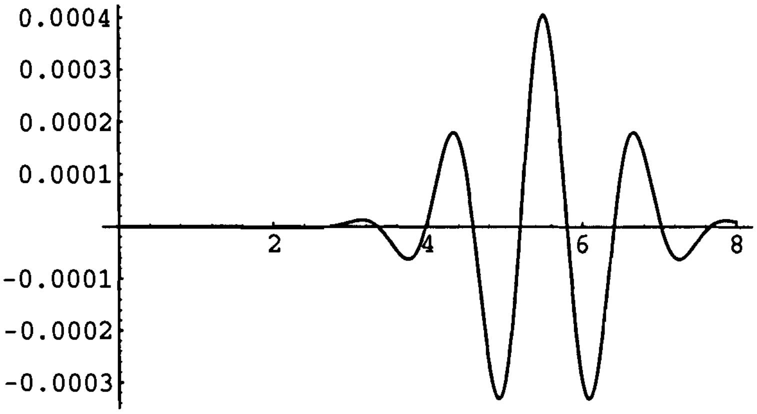

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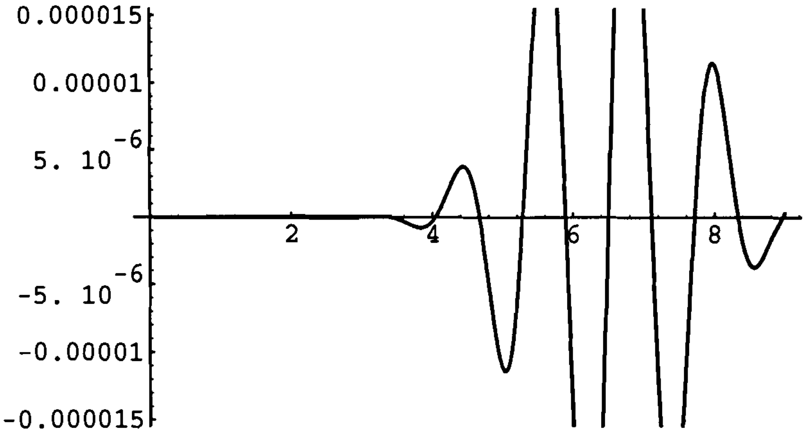

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