



Spring 4-2015

Unleashing Music's Hidden Blueprint: An Analysis of Mathematical Symmetries Used in Music (Honors)

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Unleashing Music's Hidden Blueprint

*An Analysis of Mathematical Symmetries
Used in Music*

By Natalie Hoijer

Honors Research 2015

Dr. Mario Pelusi, *Faculty sponsor*

Acknowledgements

I would like to thank Dr. Mario Pelusi and Dr. Lisa Nelson for their support and guidance as my faculty advisors through this research endeavor, as well as for allowing me to gather data and teach my findings at the Illinois Chamber Music Festival. I greatly appreciate the dedication of time, the valuable feedback and the great deal of knowledge that these professors contributed to this project. It is an honor to be an Eckley Scholar, and I am very appreciative of the funds that were made available with which to pursue this study. I feel very privileged to have had the opportunity to work with such talented faculty members on a project that I was able to initiate and organize. Both Dr. Pelusi and Dr. Nelson have offered me insightful guidance as this project developed. It has been a wonderful experience to have had unencumbered time devoted to analyzing my research, and it has been exciting to make discoveries and to put together the “pieces of the puzzle.” It has especially been a thrill to have the opportunity to share and teach my findings to the young musicians who attended last summer’s Illinois Chamber Music Festival. Receiving the students’ feedback was tremendously beneficial to my understanding of how various compositional tools can affect the performer and the listener. To transform a subjective field, such as music, into an objective understanding by utilizing colors, shapes, or proportions has been a rich and gratifying experience!

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Abstract

The history of the development of mathematics and the development of Western music unleashes fascinating connections between the two fields and illustrates their similarities and dependence on each other. Various branches of mathematics are rooted in music, ranging from mathematical physics in sound frequency, to probability and statistical methods of composing, to the use of the Golden Mean and the Fibonacci Series in music. The human brain's logical functioning left side and creative functioning right side, as studied by psychobiologist Robert Sperry ("Whole Brain Development"), are bridged together in this project as mathematical patterns meld with the art of musical composition. These studies investigate mathematical patterns such as the Fibonacci Series and the Golden Mean as they apply to the composition of concert music, in comparison to other mathematical symmetries used as compositional tools, such as palindromes, crab canons, and fractals. This research explores the impact that these compositional techniques have on the style, structure, and aesthetic beauty of a composition as a whole, and thus considers how these techniques set the piece apart from other works that do not use such mathematics. The findings show that the Fibonacci Series and Golden Mean were the most effective compositional tools and yielded the most aesthetically pleasing results.

Literature Review

Introduction

Mathematics and music have worked hand in hand throughout history. One of the more prominent individuals found in the earliest years of mathematics is Pythagoras, a Greek philosopher, mathematician, and musician, who believed that math and music provided the keys to the secrets of the world. Pythagoras was not the only dual mathematician and music theorist. In fact, it was quite common for mathematicians to be skilled in music; e.g. Archytas, Nicomachus, Ptolemy, Boethius, and Euler, just to name a few. The combination of mathematician and musician is not a coincidence, but rather, it is an indication of the close relationship that mathematics and music share. These two closely correlated fields exhibit many overlapping concepts, particularly with regard to symmetry. The phenomenon of symmetry relies on patterns, repetition, balance, and detecting invariance or change. Math and music are both substantially based on patterns and sequences. Whether symmetry is considered a geometric principle or a fundamental element of art, it is an essential component to both the sciences and the arts. Symmetry has played an invaluable role in the field of physics; e.g. through findings in quantum theory or paving the way for discoveries to be made on conservation laws. In mathematics, symmetry is the basis for geometric shapes, transformations, and graphs. With the operations of translation, reflection, and rotation, shapes are manipulated around an axis of symmetry ("Symmetry"). Similarly, symmetry is an obvious component of architecture and design throughout the world ranging from the Parthenon to the

Egyptian Pyramids. In the field of music, the use or lack of symmetry holds the same level of importance in the final outcome of the work. Composers use various forms of mathematical symmetries when creating their works, and this leads to the questions: is this method of structure an enhancement to the piece, and, if so, which type of symmetry is the most effective? Some forms of mathematical symmetries that have been used as structural tools by composers are palindromes, crab canons, and fractals. Another mathematical tool utilized by composers is the Fibonacci Series and the golden section. Interestingly, the Fibonacci Series has been studied and reported to possess an aesthetic beauty; however, this mathematical technique represents asymmetry, not symmetry. This project explores these mathematical tools as they are used as musical structures and investigates which techniques are more effective when composing a work of music.

Fibonacci/Golden Section

The Golden Mean, also known as the “Golden Proportion,” the ratio 0.618 to 1, phi, or “dynamic symmetry,” is very closely associated with the Fibonacci sequence. The Fibonacci Sequence is an infinite series of numbers that follows a pattern in which each subsequent number is the sum of the previous two numbers; i.e., {1, 1, 2, 3, 5, 8, 13, 21, 34}. If any adjacent Fibonacci numbers are divided by each other {2/3 or 21/34}, a Fibonacci ratio is formed, and as the ratios move further along in the sequence, the ratios converge to 0.618, the golden ratio (Garland, Kahn). Leonardo of Pisa, a medieval mathematician, also known as Fibonacci, advanced the development of mathematics in Europe by publishing a book in 1202 titled *Liber Abaci*, in which he introduced Arabic numbers. He is even more famous for his contribution of creating a

number sequence that was needed to solve a hypothetical problem of breeding rabbits. This sequence, which later came to be known as the Fibonacci sequence, bridged the knowledge of the golden mean from the Pythagoreans and has inspired great achievements in architecture and sculptors from the Greeks up to modern day (Madden). This sequence has been a curious topic of study for mathematicians, musicians, artists, botanists, and astronomers alike, as it is found in nature with the spiral of flower petals, the architecture of buildings, proportions of the human body, and even in the design of the piano keyboard. On a piano keyboard, there are eight white keys that span an octave; e.g., from C_4 to C_5 . There are five black keys within that octave, separated into groups of two keys and then three keys. All together there are thirteen white and black keys within the octave, and those numbers, {1 octave, 2 black, 3 black, 5 black, 8 white, 13 total} are the first six numbers of the Fibonacci series. The violin is another example of an instrument that embodies the Fibonacci numbers, as the architecture of the instrument possesses the golden proportion. The structure of the violin is proportioned so that the length of the body compared to the length of the fingerboard forms a golden ratio. There are countless other intriguing applications of this curious sequence (Garland, Kahn)!

This specific ratio, known as the Golden Proportion, has an aesthetic appeal that creates a sense of beauty and balance. From a visual perspective, many studies show that the golden proportion offers the most pleasing display at which to look. According to Adrian Bejan, a mechanical engineering professor at Duke University, “the human eye is capable of interpreting an image featuring the golden ratio faster than any other” ... “Whether intentional or not, the ratio represents the best

proportions to transfer to the brain” (McVeigh). Bejan claims that animals and humans are oriented in the horizontal. They absorb information more effectively when they scan side to side, and shapes resembling the golden ratio aid in the scanning and transmission process of the vision organs to the brain. Analyzing the appeal of the golden ratio from a scientific standpoint, Bejan states that animals are wired to feel more satisfied when they are assisted, so since the golden ratio proportion helps the brain process an image, the result is feeling pleasure that is translated into beauty (McVeigh). Likewise, from an aural perspective, the golden proportion offers the same satisfaction in the form of sound. For instance, specific chords utilize the Fibonacci ratio, such as major or minor sixth chords, which interestingly are considered to be the more pleasing intervals. A major sixth interval consisting of C and A entail 264 vibrations per second for the C and 440 vibrations per second for the A. This ratio, $264/440$ simplifies to $3/5$, which is a Fibonacci ratio. Similarly, a minor sixth interval of E and C produces a ratio of 330 vibrations per second to 528 vibrations per second, equivalent to $5/8$, another Fibonacci ratio. Any sixth interval reduces to a similar ratio of vibrations (Garland, Kahn).

The Fibonacci series and golden proportion phenomenon have been incorporated into music throughout the different eras of music. Composers such as Béla Bartók and Claude Debussy, aware of this influence of the golden proportion, used the Golden Mean or the Fibonacci Sequence as a compositional tool in their music, particularly in regard to form (Garland, Kahn). Bartók’s works utilize the golden section in proportions of lengths of movements, main divisions of a composition, and even chordal structures, as shown by Erno Lendvai in his book, *Béla*

Bartók, An analysis of his music. Bartók created these types of progressive mathematical works in his final period, which was around 1926 to 1945 (Antokoletz). In Bartók's *Sonata for Two Pianos and Percussion*, the first movement consists of 443 measures. Lendvai explains that in order to easily calculate the golden section of a piece of music, find the product of the total number of measures of a work and the ratio phi. Thus, in this particular movement, the formula would follow $443 \times 0.618 = 274$. Measure 274 marks the golden section of this movement, and it is precisely in this measure that the recapitulation begins. Similarly, in the 93-measured movement I of Bartók's *Contrasts*, the recapitulation starts in measure 57 (93×0.618), the golden section of the movement.

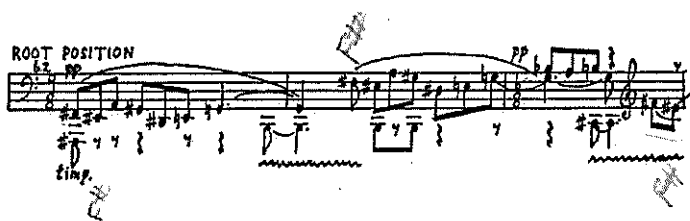


Figure 1a (Lendvai, 18)

On a more local level, *Sonata for Two Pianos and Percussion* demonstrates golden section construction within the first 17 measures of the piece. The first group of measures (mm. 2-5) is in the area of the tonic, the second group (mm. 8-9) is in the dominant, and the third, (mms. 12 and on) is in the subdominant, and it is also an inversion of the first two sections (see Fig. 1a and 1b).

Figure 1b (Lendvai, 19)

The entire form is comprised of 46 units of $3/8$ time (with $9/8$ time signature). $46 \times 0.618 = 28$, and 28 units cover the part right up until the inversion. Thus, the beginning of the inverted section is marked by the golden section: 28. The golden section is seen to coincide with the more significant turning points in the structure (see Fig. 2 and 3).

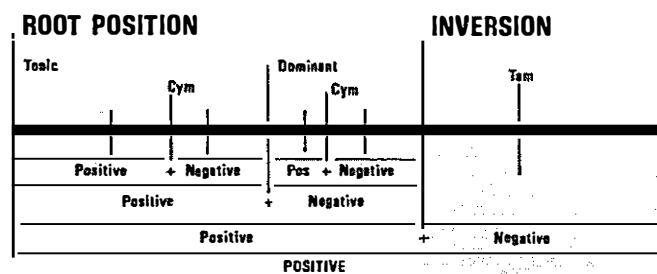


Figure 2 (Lendvai, 20)

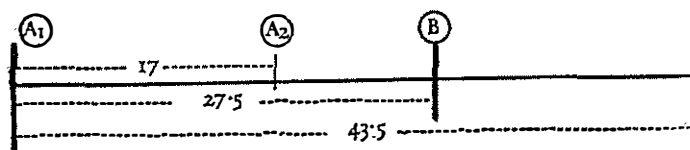


Figure 3 (Lendvai, 22)

The third movement is structurally proportioned by the golden ratio as well. The principal theme, 43.5 measures in length, is divided into three sections: $A_1 + A_2 + B$. The location of the B section is dictated by $43.5 \times 0.618 = 27.5$. Likewise, the A sections also have a phi relationship with each other. Both A sections together total to 27.5 measures, and $27.5 \times 0.618 = 17$, thus making A_1 17 bars in length (see Fig. 3). Furthermore, particular masterpieces of Bartók's go beyond the use of the golden section and include Fibonacci numbers. The numbers that make up the mystifying Fibonacci sequence appear in the framework of the first movement of Bartók's *Music*

for Strings, Percussion and Celesta. The movement, which is 89 measures in length, climaxes at bar 55, thus dividing the movement into a section of 55 measures and a section of 34 measures. Not only is the peak that occurs at bar 55 the golden section of the movement, all of the durations created here (89 bars, 55, bars, and 34, bars) are also Fibonacci numbers. The climax at bar 55 is emphasized further by dynamic levels, as the movement starts *pp*, the crescendo grows throughout the movement, until it reaches *fff* at bar 55, and then diminishes to *ppp* at the conclusion of the movement. As illustrated in Fig. 4, the segments of music continue to be subdivided into golden proportions within the structure of the piece; e.g., the 34 bar section following the 55 bars and the climax, is divided into 13 and 21 bars. Again, 13 and 21 are both Fibonacci numbers, and they form a golden ratio ($34 \times 0.618 = 21$ and $34 - 21 = 13$). Since, the smaller portion of the ratio appears first (the 13 bars) and is followed by the longer portion (the 21 bars), this is an example known as the inverse golden section (Lendvai). Phi is unique to other ratios in that its inverse is itself minus 1 (Madden).

$$\frac{1}{\phi} = \phi - 1$$

$$\frac{1}{1.618} = 1.618 - 1$$

$$0.618 = 0.618 \quad (\text{Madden})$$

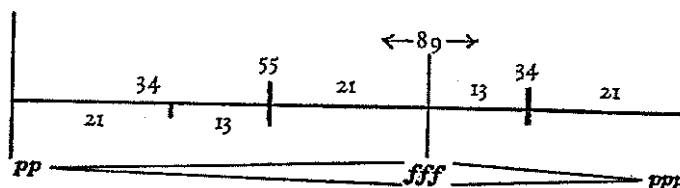


Figure 4 (Lendvai, 28)

The divisions of the piece into golden sections and inverse golden sections fit together and lead into the next, creating a natural, wave-like flow. The continuous pattern forms, both large scale and small scale, rises and falls. According to Lendvai, “‘We follow nature in composition,’ wrote Bartók, who was indeed directed by natural phenomena to his discovery of these regularities. He was constantly augmenting his collection of plants, insects, and mineral specimens (Lendvai).” Clearly, Bartók implemented his fascination for nature into his music. The fact that he was aware of the presence of Fibonacci numbers in nature, such as tree branches showing the yearly increase according to the Fibonacci sequence, the spiral of sunflower seeds, the pattern of pine cones that correspond to Fibonacci numbers, and flowers that possess Fibonacci-numbered petals, demonstrate that Bartók’s use of such patterns in his music was intentional. He even made a note that sunflowers were his favorite plant, and he was always brought joy with fir-cones (Lendvai).

Debussy is another composer known for utilizing the golden section in his compositions. In his piano work, *Images (1905)*, he structures the first movement, *Reflects dans l’eau*, with the golden proportion. *Images* is a pivotal work in Debussy’s career, as it is a foundational anchoring for Debussy’s development of music possessing proportional patterns. In *Reflects dans l’eau*, the 94 measure movement

consists of growing wave-like features, similar to the Bartók pieces discussed earlier. This wave-like property is enhanced through the dynamic markings indicated in this movement. The work begins at *pp*, grows to a crescendo, and climaxes at *ff* in m. 58, after which a decrescendo begins until finally reaching *ppp* at bar 94. The golden section of the 94 measures ($94 \times 0.618 = 58$) occurs in m. 58, which is precisely where the dynamics peak. The ratio of 58/94 not only resembles the golden ratio, but it also illustrates two numbers from the Lucas Series. The Lucas Series is an adaptation of the Fibonacci Series and exhibits similar properties, but it begins with the number 3, as opposed to beginning with the number 1. Similarly, the Lucas series unfolds as follows: {3, 4, 7, 11, 18, 29, 47...}. 58/94 reduces to 29/47, which is a Lucas ratio, and all Lucas ratios converge to 0.618 since they have the same properties as the Fibonacci series (see Figure 5). Also, the first entrance of the beginning episode of the piece occurs at m. 23, which is the inverse golden section of the first 58 bar segment ($58 \times 0.618 = 35, 58 - 35 = 23$). Similarly, the final exit of the coda occurs at m. 80, which is the golden section of the last 36 measures, following the *ff* peak at m. 58 ($36 \times 0.618 = 22, 58 + 22 = 80$). These golden proportions within golden proportions can be observed in the diagram in Figure 5 (Howat).

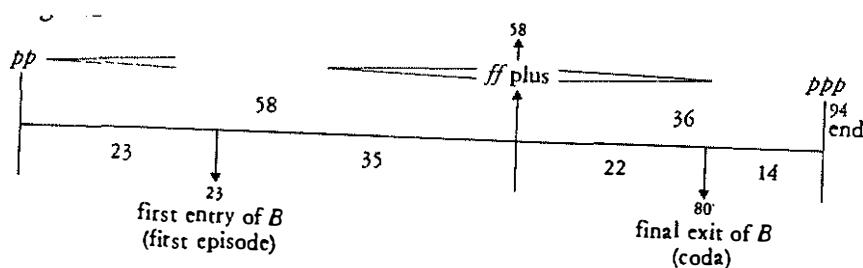


Figure 5 (Howat, 25)

It is interesting to note whether these patterns and proportions were written intentionally or intuitively. The answer to this question can vary from composer to composer, but as for Debussy, most of his compositions that exhibit golden section proportions were created that way deliberately. Debussy's awareness of the golden section is evident because he had constant associations with painters and other artists who were avid users of the golden section. The interest in applying the golden section in the visual arts was endemic, as documented by the exhibition in Paris in 1912 by the Section d'or (golden section) group of painters. Among these French Symbolist artists, applying proportional techniques to art was a well-known concept. In addition to Debussy's connections to these artists, his personal taste for composition pointed toward his awareness of using the golden proportion. It is known that Debussy disliked musical formulas; i.e., prescribed recipes, so to speak (e.g. sonata form and fugues). "By contrast, GS [golden section] is a natural principle, like the harmonic series, whose physical existence antedates mankind, (Howat, 9)" states Roy Howat in his book *Debussy in Proportion*. "When he [Debussy] wrote, more than once, about his musical 'search for a world of sensations and forms in constant renewal,' his aim was evidently to free music from rigidly stereotyped forms" (Howat, 9). The most concrete evidence of Debussy's conscious work with numbers is a letter from August 1903 to his publisher, Jacques Durand, which referenced corrected proofs of his composition *Estampes*. This letter refers to a missing measure that is not in the manuscript, but was necessary due to the divine number, which is another name for the golden number or golden proportion. These examples provide some evidence that Debussy did engage, to some extent, in the use of the golden section in his music.

It is not certain if he was conscious of his use of the golden section at all times, but even the use of these patterns subconsciously as a composer offers an intriguing exploration. Composers who may not have been familiar with the golden section or who did not purposely incorporate the idea into their music have produced musical works that do appear to use such techniques. Analyses have been published on works by Beethoven and Bach in which it seems as if Fibonacci numbers and golden-proportioned climaxes are apparent, but without evidence that the use of these techniques was deliberate (Howat). This fact of intuitively placing a climax at 0.618 or 61.8% of the way through a composition emphasizes the natural appeal that this proportion has for humans. This natural inclination to peak at this ratio brings back the study that Bejan explored with the golden ratio related to visual aesthetic beauty, and thus it raises the question: does this intuitive placement of the climax indicate an aural beauty?

“Anything that is divided along this proportion is not static, even, or geometrically symmetric. Rather, it seems to have a flow to it, or a quality known as dynamic symmetry (Garland, Kahn).”

Palindromes

In contrast to the Fibonacci series and the golden section as compositional tools, palindromes yield a true symmetrical pattern. A palindrome is a word, phrase, number, or sequence that is the same both forwards and backwards. This term, coined in the 17th century by English writer Ben Jonson, combines the Greek roots “palin,” meaning “again,” and “dromos,” meaning “direction”

(<http://www.palindromelist.net>). Common words such as “wow” or “racecar” are palindromic as are numbers like 343 or 1,024,201. On a more complex level, composers, such as Haydn, Hindemith, Berg, and even going as far back as Machaut, have used palindromes as a compositional device, from a pitch perspective, from a rhythmic perspective, or from a pitch and rhythmic perspective.

One very clear example of a musical palindrome is found in Haydn’s *Symphony No. 41 Movement III: Menuet and Trio*. The entire third movement of the symphony is a palindrome. For example, with regard to form, it is Menuet, Trio, Menuet, and on a more detailed level, the pitches and rhythms are identical forwards and backwards in the oboes, horns, violin I and violin II parts. Starting with the very first note at the beginning and the very last note at the end and working inward, one can observe that the notes match up exactly for the twenty measures of the Menuet at the start and finish of the movement with the twenty measures of the trio in the center of the movement. Aside from the lack of repeats in the Menuet, the second time the Menuet enters, there are no other deviations in those top voices (see score in Figure 6).

The image displays a page of musical notation for Hindemith's *Ludus Tonalis*, specifically the third movement, *III*. The score is organized into three main sections: *Menuet*, *Trio*, and *Menuet [da capo]*. The *Menuet* section is marked with a tempo of *allegro* and a dynamic of *mf*. It includes staves for 2 Oboes, 2 Cori in C/Sol, Violino I, Violino II, Viola, and Violoncelli, Basso e Fagotto. The *Trio* section is also marked with a tempo of *allegro* and a dynamic of *mf*, featuring staves for Violino I, Violino II, and Violoncelli, Basso e Fagotto. The *Menuet [da capo]* section is marked with a tempo of *allegro* and a dynamic of *mf*, featuring staves for Violino I, Violino II, and Violoncelli, Basso e Fagotto. The score is framed by large grey arrows pointing outwards from the corners.

Figure 6

In Hindemith's *Ludus Tonalis*, the musical palindrome utilized is one that illustrates large scale structural symmetry. The fugues and interludes that comprise this work are framed by a Praeludium in the very beginning and a Postludium at the very end of the piece. The Praeludium (mm. 1-49) is a direct palindrome of the Postludium; in other words, the Praeludium in reverse is the Postludium, excluding the final bar of the Postludium. The palindromic properties persist in terms of pitch class, not necessarily in the same octave, and rhythm. The top staff creates a palindrome with the bottom staff and vice versa. Additionally, the interludes are placed around the center *March*, and the interludes are arranged stylistically such that the styles of the alternating interludes form a palindrome. To elaborate, on either side

of the *March* is a Romantic miniature, and both before and after the Romantic miniature is a Baroque interlude. Therefore, the styles form a palindromic pattern: Baroque, Romantic, March, Romantic, Baroque. Thus, Hindemith demonstrates big picture structural symmetry through his use of palindromes (See Figure 7).

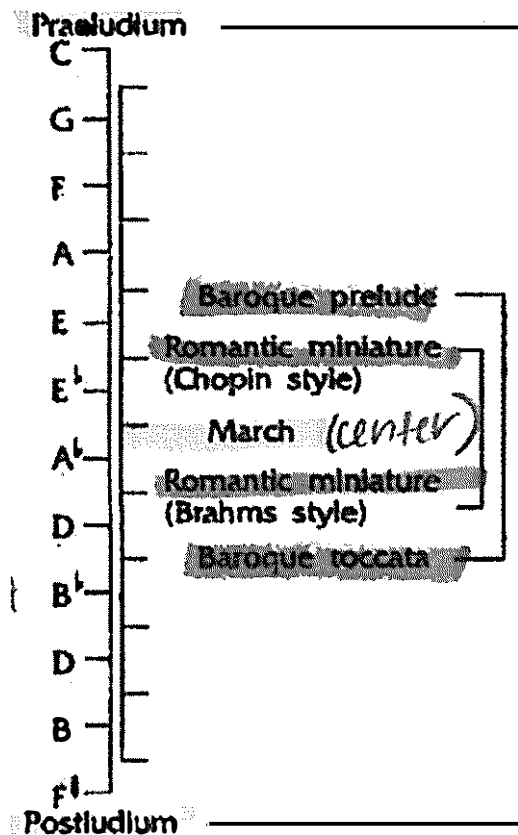


Figure 7 (Bruhn)

In Act II of Berg's opera *Lulu*, the sequence of events form an intriguing palindrome centered around a fermata that acts as the center of the scene. This palindrome occurs in the short film portion of Act II, and the level of detail of how the events correlate with other events equidistant from the fermata is amazing! For instance, the 35 bars before and after the fermata (m. 687) correspond thematically

with each other. Specifically, seven bars before the fermata “the door shuts” while seven bars after the fermata “the door opens.” Nineteen bars before the fermata is “dwindling hope,” while nineteen bars after the fermata is “growing hope,” and twenty-three bars both before and after the fermata is “in nervous expectation.” (<http://www.ibiblio.org/johncovach/bergtime.htm>) (See Figure 8 for more details)

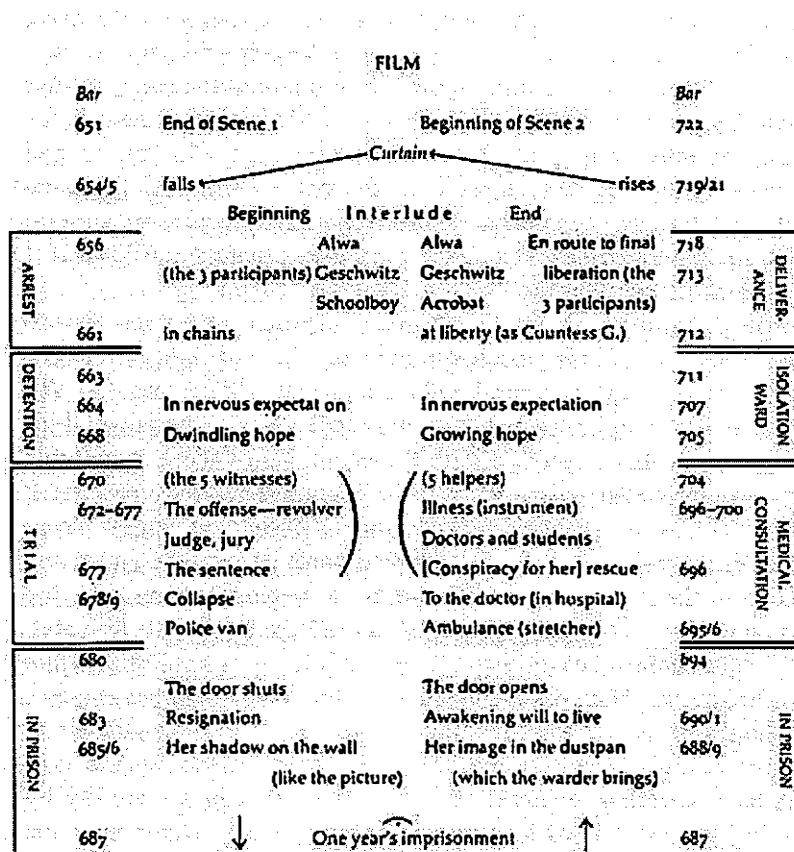


Figure 8

Dating back to as far back as the Renaissance, palindromes have been found in works by Machaut. The composition *Ma fin est mon commencement*, which translates as *My end is my beginning*, was written by Machaut around the 1360's and clearly

displays his intent to create a palindrome, since the title of the work and the text explain the palindromic structure of the music. As the title exclaims, the end is the same as the beginning, in reverse. Of the three voices, the Triplum and the Cantus are palindromes of each other, so the beginning of the Triplum and the end of the Cantus can be traced back note by note, reaching the double bar directly in the middle of the piece at bar 21. Similarly, the beginning of the Cantus corresponds to the end of the Triplum. Then, the third voice, the tenor, is a palindrome with itself, thus it is the same forwards and backwards.

Crab Canon

Another form of symmetry used as a compositional tool in music is the crab canon. Similar to the palindromes, crab canons rely on symmetry and patterns, but crab canons are not simply the same forwards and backwards. Crab canons incorporate another dimension in which a canon is made up of two complementary reversed lines of music. Often crab canons are associated with Möbius strips, as the paradoxical shape of Möbius strips provide a visual representation for this highly complex pattern. J.S. Bach is the creator of one of the more stunningly impressive compositions to include a crab canon, namely, *The Musical Offering*. In this piece, Bach wrote a melody, then reversed the melody to be played note by note in the opposite direction, then he added another layer in which the forwards and backwards melodies are played simultaneously, and finally, he added a crab canon to crown the entire work. There is a short video clip, created by Jos Leys, a mathematical image creator, that illustrates these structures (<https://www.youtube.com/watch?v=xUHQ2ybTejU>).

Mirroring is another form of symmetrical composition in which the music is repeated in an inverted variation, rotated, or flipped upside down. Mozart used this technique in his composition *Der Spiegel*, also known as the *Mirror Duet* for two violins. This duet is unique in that it is single sheet of music from which both musicians perform (by standing on opposite sides of the stand that is laying flat for both violinists to see). To clarify, the first violinist reads the music from top to bottom, while the other violinist, standing on the opposite side, reads the music from top to bottom (inverted staff), which is really the first violinist's bottom. The cleverness and thought that went into this composition are evident in the fact that the two voices harmonize with each other just like they would in any other duet, but Mozart was limited to the restrictions of writing both parts upside down on a single inverted staff, with clefs running down both sheets of the paper. Moreover, while the first note is read as a D for the first violinist, this very same note is a G for the second violinist's final note. Each chord and phrase harmonizes and fits together like the pieces of a puzzle. The score can be viewed in the Appendix.

Fractals

Another mathematical symmetry that has application in music compositions is the concept of fractals. A scaling fractal is an infinitely self-similar geometric object. By magnifying a fractal over and over again, the result is the same pattern or shape at different scales. Natural fractals surround us in nature, such as snowflakes, or the veins in leaves branching in self-similar proportional patterns. The notion of self-similarity has, for centuries, been incorporated into musical compositions. On a

very broad level, dividing compositions into more movements and basing a work on a motive are examples of this parallelism between fractals and composition.

However, a more detailed correlation of fractals and music is the transformation of motives and themes through augmentation, diminution, transposition, retrograde, inversion, etc. These various techniques generate musical fractals, because a single musical theme that is altered proportionally produces a phrase that resembles the original; i.e., it is self-similar to the original but is longer, shorter, or shifted over (Garland, Kahn).

In Bach's "Contrapuntus VIII," a musical fractal can be identified in the construction of harmonies among the different instruments. The horn plays the main theme, and that theme, with a slight alteration, is passed on to the other instruments (Figure 9). The first trumpet has the same theme flipped upside down, so the rhythms are consistent but the pitches are inverted (Figure 10). The second trumpet has the same pitches of the inverted first trumpet's part, but the rhythmic deviations are longer; this is known as rhythmic augmentation. The tuba has a more extreme version of the augmented melody. The durations of the rhythms are held proportionally longer than the original theme. All of the instruments are then harmonized concurrently, and they proceed at their own pace, comprising the *Contrapuntus VII* (Figure 11). An animated video clip, created as a visual aid by Stephen Malinowski, illustrates clearly the shapes of these musical fractals and how they overlap each other when the whole piece is put together (see link for access to video clip: <https://www.youtube.com/watch?v=V5tUM5aLHPA>).



Figure 9



Figure 10

Alterations or operations that generate musical fractals can be seen in different methods of composition; e.g., twelve-tone music, serial music, and serial twelve-tone music. In the early twentieth century, Arnold Schoenberg, Alban Berg, and Anton Webern developed a method of composing nontonal music that was based on patterns of pitch classes that contained all twelve pitch classes of the chromatic scale in which the pitches are assigned numbers (0=C, 1=C#, 2=D, 3=D#). Specifically, Schoenberg's method consisted of using a particular ordering of the twelve pitch classes, known as the row, or set, as the source material for a composition. He would then manipulate the set by using a variety of operations; e.g., transposition, inversion, and retrograde. Transitions are one of the operations that can be performed to a set or to a tone row (Loy). The notation that musicologists use to portray "transposition by n semitones," is " T_n ." For example, T_2 represents transposition by two semitones. Since the chromatic scale has twelve pitches, the operations all follow modulo 12, thus T_{12} is equivalent to T_0 , which is no transposition. Similarly, T_{14} is equivalent to T_2 , because $14 \bmod 12 = 2$ (Harkleroad). The definition of transposition is $T_n(x) = ((x + n))_{12}$, where x represents the pitch class that is being transposed and n represents the number of degrees that it is being transposed. For instance, $T_4(\{4, 6, 7, 10\}) = \{8, 10, 11, 2\}$, in

which T_4 is the notation for a transposition of four. Hence, each value in the set is increased by four, and to clarify, each value in the set represents a pitch in the chromatic scale (C=0, C#=1, D=2...etc.) (Loy).

Inversion is another operation that is applied to sets. By subtracting each pitch class from the number of elements it has, a mirror image is created, and this is known as inversion. It is defined as $(I(x) = ((N - x))_N)$, in which N is equal to 12, the number of available pitch classes. The inversion operation being applied to the set {4, 6, 7, 10} would look like $I(\{4, 6, 7, 10\}) = \{8, 6, 5, 2\}$. One last operation is retrograde. The retrograde operation reverses the order of the members of the sets. So, $R(\{4, 6, 7, 10\}) = \{10, 7, 6, 4\}$ (Loy). These three operations: transitions, inversions, and retrogrades, can be combined to form more complex operations, such as $T_n, T_nI, T_nR,$ and T_nIR . With the different combinations of operations; $T_n, T_nI, T_nR,$ and T_nIR , along with the twelve choices for n , there is a total of forty-eight possible operations (Harkleroad).

Conclusion

My investigation has focused on mathematical patterns such as the Golden Mean as it applies to the composition of classical music, in comparison to other mathematical symmetries used as compositional tools, such as palindromes, crab canons, and fractals. This research explores the impact that these compositional techniques have on the style and structure of the piece as a whole, and thus considers how these techniques set a composition apart from other works that do not use such aspects of mathematics.

Research Confirmation Results

According to Gareth E. Roberts at the College of Holy Cross, who references John Putz from *Mathematics Magazine*, Mozart's piano sonatas possess the golden mean in the structural divisions of the pieces (http://www.americanscientist.org/issues/pub/did-mozart-use-the-golden-section). Upon becoming aware of this belief, the scores for these piano sonatas are obtained and analyzed to determine if Putz's claim is verifiable. From analyzing three of the sonatas listed below, Sonata 279, 280, and 283, it can be deduced that there some truth to this claim.

TABLE 1

Köchel	<i>a</i>	<i>b</i>	<i>a + b</i>
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	38	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102
283, I	53	67	120
283, II	14	23	37
283, III	102	171	273
284, I	51	76	127
309, I	58	97	155
311, I	39	73	112
310, I	49	84	133
330, I	58	92	150
330, III	68	103	171
332, I	93	138	229
332, III	90	155	245
333, I	63	102	165
333, II	31	50	81
457, I	74	93	167
533, I	102	137	239
533, II	46	76	122

<http://mathcs.holycross.edu/~groberts/Courses/Mont2/Handouts/Lectures/Mozart-web.pdf>

Figure 11a

Confirmation:

Sonata 279

Mvt 1: $100 \times 0.618 = 61.8 \approx 62$	✓
Mvt 2: $74 \times 0.618 = 45.73 \approx 46$	✓
Mvt3: $158 \times 0.618 = 97.64 \approx 98 \neq 102$	4 off

Sonata 280

Mvt 1: $144 \times 0.618 = 88.99 \approx 88$	✓
Mvt 2: $60 \times 0.618 = 37.08 \approx 37 \neq 36$	1 off
Mvt3: $190 \times 0.618 = 117.42 \approx 117 \neq 113$	4 off

Sonata 283

Mvt 1: $120 \times 0.618 = 74.16 \approx 74 \neq 67$	7 off
Mvt 2: $37 \times 0.618 = 22.87 \approx 23$	✓
Mvt3: $273 \times 0.618 = 168.71 \approx 169 \neq 171$	2 off

Figure 11b

Although not all of the movements of the sonatas follow the golden proportion exactly, many of them do, and the others are all within close proximity and within range of the golden section.

As seen in the confirmation displayed in Figure 11b, four of these nine movements are divided by a double repeat sign exactly at the golden section, more precisely at the inverse golden section, because the smaller proportion of the ratio occurs first in these movements. For instance, Sonata 279 Movement 1 has a double repeat sign following measure 38, leaving 62 bars remaining ($100 \times 0.618 = 61.8 \approx 62$, $100 - 62 = 38$). The other movements also have double repeat signs that are very close to the inverse golden section, only 1, 2, 4, or 7 bars off, which is a very small error. That small discrepancy is still within range of the inverse golden section.

After analyzing a plethora of “classical” pieces that utilize the golden section and Fibonacci numbers, I was inspired to explore other genres of music such as familiar tunes and pop songs. For example, our national anthem, the *Star Spangled Banner*, climaxes at the golden section of the piece. Specifically, in a 75 second recording of the *Star Spangled Banner*, the peak at which the climatic, “Oh say does that star spangled banner yet wave...” is sung 46 seconds of the way through ($75 \times 0.618 = 46.35$). Thus, 46 seconds is the golden ratio of the 75-second song and the climax falls exactly on this location.

Additionally, the golden section is utilized in pop songs. While examining various popular songs for this mathematical technique, the golden section was found to have a significant location in the Beatles hit *In my Life*. This Beatles tune is 144 seconds long, so the golden section occurs 89 seconds of the way through (144

$\times 0.618 = 88.992$). At precisely 89 seconds through, an instrumental break interrupts the flow of the song with a dramatic difference in timbre. The instrumental break is played on harpsichord, which drastically contrasts with the voice and guitar that are heard up to this moment. It is interesting that a significant transition and a change of instrumentation that surely attracts the listener's attention occur at the golden section.

As mentioned previously, it cannot always be determined if the golden mean is used intentionally or if the climax happens to coincide with the golden section by chance. However, either scenario leads to a compelling point. If the Beatles were aware of the golden section and believed that it was the most aesthetically pleasing place, it is logical that they would place the significant turn in the song at that location. If the Beatles were unaware of this 0.618 proportion, and their instrumental break coincidentally fell exactly 61.8% of the way through the piece, then this supports the notion that humans are naturally attracted to this proportion.

One other genre of music in which Fibonacci numbers have been found is in film music, specifically action music. Lalo Schifrin, composer of film scores for *Mission: Impossible*, *Mannix*, *Starsky and Hutch*, and the *Dirty Harry* films, just to name a few, states in his book, *Music Composition for Film and Television*, that unpredictable rhythms enhance a cue's sense of excitement in an action film. Fibonacci numbers, although possessing an organic relationship to each other, are unpredictable. Therefore, the use of Fibonacci numbers can create an unpredictable sound, while also maintaining organization. For example, the Fibonacci numbers may be applied to the music in regard to rhythm too, such as groupings of accented

triplets. As shown in Figure 12 (an excerpt from Schifrin's book), the groupings of triplets in the top staff follow the numbers listed in the Fibonacci series (1, 1, 2, 3, 5, 8, 13). First there is one set of triplets, then after a quarter rest there are 2 sets, then 3 sets, then 5 sets, then 8 sets, then 13 sets. This allows enough unpredictability in listening to the score, yet also possesses a visible, mathematical pattern while analyzing the score.

ca 144
Fibonacci

(1) (2) (3) (5)

(4) (5) (6)

(7) (8) (9) (10)

etc.

Figure 12 (Schifrin)

Another piece that illustrates palindromes in music is Webern's *Symphony Op. 21*. In the second movement, within the first eleven measures, there are rhythmic, intervallic, and dynamic palindromes. The rhythms are consistently palindromic, centered around the two eighth notes in measure 6, and they move at the same intervals, thus creating a rhythmic and intervallic palindrome. For instance in both measure 1 and measure 11, there are two quarter notes that are held at a constant pitch, then in bar 2 and bar 10, working inward toward the center, there is rest and a leap up a third to another quarter note. As shown in Figure 14, the rhythms and intervals continue to match up in the bars equidistant to bar 6.

The image shows a musical score for Webern's *Symphony Op. 21*, second movement, measures 1 through 11. The score is for Clarinetto, Corni I and II, Arpa, Violini I and II, Viole, and Violoncelli. The tempo is marked "Sehr ruhig" with a metronome marking of a quarter note equal to approximately 54 beats per minute. The score illustrates a rhythmic and intervallic palindrome centered on measure 6. Brackets above the Clarinetto staff indicate the palindromic structure. Dynamics include *pp*, *p*, and *con sord.* (con sordano). The Clarinetto part has notes in measures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. The Corni parts have notes in measures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. The Arpa part has notes in measures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. The Violini, Viole, and Violoncelli parts have notes in measures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11.

Figure 14

In addition to observing the palindromes found within the rhythm and pitches, it is interesting to note the palindrome that occurs below the staff. The dynamics through these 11 measures create a palindrome pattern, such that m. 1 starts *pp* just as measure 11 ends *pp*; m. 2 is *p*, which matches m. 10 at *p*; and then m. 4 has a decrescendo to *pp* in m. 5 and then a crescendo into to m. 6, just as in mss. 7, 8, and 9. These examples of palindromes are instances of small-scale symmetry, as opposed to Hindemith's *Ludus Tonalis*, in which large-scale palindromes are used in the ordering of the movements. This variety of symmetry illustrates that musical palindromes can be found on both a small scale and a large scale, whether it is throughout an entire composition, or only in sections of a work.

Class Procedure

The following corresponds with lesson plans that were presented to a classroom of twenty to thirty high school students over the course of ten, forty-five minute sessions spread out over a three-week long music camp.

The audience for this study consisted of young musicians who attended the Illinois Wesleyan Chamber Music Festival in the summer of 2014, and their ages ranged from eighth grade to freshman in college. These lessons were taught as the camp elective course (from 6:30-7:15p.m. in Presser Hall).

Day 1

On the first day, students introduced themselves and stated a number between 100 and 9,999 that represented themselves. The names and numbers were written on a piece of butcher paper at the front board and was hung in the classroom until the significance of the numbers were revealed at the end of the camp.

Then the students participated in a comparative listening activity. Two anonymous pieces were played for the students from *Music for Strings, Percussion and Celesta* Movement 1 by Bela Bartok, which represents a piece that utilizes the golden section and the hymn, *The Faith of our Fathers*, which represents a piece that does not utilize the golden section. First the Bartok was played while the students just listened, then the second time through, the students painted what colors, shapes, and imagery they heard depicted in the piece. This exercise was repeated for *The Faith of our Fathers* hymn.

Day 2

On the second day, students began class by completing “The Line Test” activity. Each student was given a slip of paper with a 16cm horizontal line and they were to place a hash mark or a dash intersecting the line wherever their attention was drawn to most, wherever their eyes were focused to, or wherever they found it most aesthetically pleasing. The students were not told anything about the golden section at this time. Once their individual slips of paper had been collected, the students came to the board and marked their hash mark on the enlarged line on the

butcher paper. This allowed the students to visually compare their hash mark placement with their classmates'. Next, the students presented their paintings from the previous day, while I took notes of the class's comments on large butcher paper posters.

Day 3

(Before this class, the student's paintings were divided by asymmetrical Piece 1 paintings, asymmetrical Piece 2 paintings, symmetrical Piece 1 paintings, and symmetrical Piece 2 paintings. The asymmetrical paintings were hung on one wall of the classroom and the symmetrical paintings were hung on the opposite wall.)

On the third day, the information on the golden section was revealed and debriefed. First, the class anonymously listened to *Reflects dans l'eau* by Debussy, as this is another example of a piece that utilizes the golden section. While the student's were listening to this recording, they completed the "Quality of a Piece Survey" (See Appendix). On the back of the survey, students were asked to observe the paintings and try to figure out why the paintings were grouped or separated the way that they were.

Next, students took the "Paintings Survey" (See Appendix) in which they chose one of two pictures that they found more aesthetically pleasing. Students went through this process for the six pairs of pictures provided. Then, as a class the pairs of paintings were discussed and the students move to the left or the right side of the room, according to which of the two paintings they selected. In every pair of

paintings, one image utilized the golden section, whereas the other used symmetry. Following this activity, the results of “The Line Test” were presented. Now that the class was aware of the presence of the golden ratio, a discussion ensued on the student’s paintings from the first class, which led into a discussion of the Debussy piece that began the current class session.

Day 4

The fourth class began by opening a discussion on the application of the golden ratio in every day life and a short video clip that elaborated on these applications (<https://www.youtube.com/watch?v=0tAZe6pP-FM>). Then, the class was organized into groups of three to work on the Mozart Sonata Activity. For this activity, each group was assigned two movements from Mozart’s *Sonata 279*, *Sonata 280*, or *Sonata 283*. The students were asked to number the measures, and calculate the golden section and the inverse golden section. Then the groups found the locations of the golden section and inverse golden section and determined if anything significant occurred at these places. At the end of class, the class regrouped as a whole to share discoveries and also discussed the prevalence of the golden section in pop music. Examples of pop tunes that utilize the golden mean were presented to the class, such as *In my Life* by the Beatles.

Day 5

The fifth day was Fun with Fibonacci Day. Now that the students were aware of Fibonacci and golden proportion tools used in music, this day offered the opportunity to explore these techniques in applications outside of music. The main

activity of the day was the Da Vinci Golden Section Body Proportion Exercise. In this activity, the students paired up and measured various body lengths of their partners and calculated the ratio of the proportions. The students measured and calculated the following ratios for their partner:

1. Height and navel height
2. Length of index finger and the distance from fingertip to first knuckle
3. Length of leg and the distance between hip and kneecap
4. Length of arm and the distance from fingertip to elbow
5. Length from top of head to bottom of chin and the length of the bottom of the ear to the bottom of the chin

The ratio is calculated by taking the first body part that is listed (for example height) and dividing it by the second body part listed (navel height). The ratio should be near the ratio 1.618. This ratio is 1.618, opposed to 0.618, because in this exercise, the larger measurement is being divided by the smaller measurement.

(See handout in Appendix for more detail)

Day 6

The sixth day began the exploration of palindromes in music. The students listened to *Ma fin est mon commencement* by Machaut, but the students were not told what they were listening to. During this listening, they completed the “Quality of the Piece Survey” (see in Appendix), as they did for the Debussy listening. Then palindromes were introduced to the students and they brainstormed any palindromes that they were already familiar with. Next, the students performed an

ice breaker activity to match the theme of palindromes. For this activity, the students each received an index card with a number printed on it and they got themselves in line, forming a palindrome without speaking in the process. The first time, the class was divided into three smaller groups, and then the second time, the whole class performed the same exercise. After the icebreaker, the class began learning the clapping Palindrome Routine Part 1 for the Camp Talent Show. The Palindrome Routine Part 1 was comprised of five sections in which the class stood in an arc that was divided into five sections. Each group was assigned one of the five sections. Each of the five sections formed a palindrome rhythmically within itself, and the five sections formed a larger scale palindrome as a whole. More specifically, section one was identical to section five, section two was identical to section four, and section three was the center of the palindrome (see music for Palindrome Routine in Appendix).

Day 7

On day seven, the students began with a memory card game based on the Palindrome found in Berg's opera, *Lulu* Act II. The class was organized into groups of five, and each group was given a set of Lulu Memory Game index cards. As a group they took turns pairing up cards, and then the next task was to place the events on the cards in chronological order, around the fermata that is located at the center of the palindrome (see image of final product Lulu activity in the Appendix).

For the second half of class, the students learned the Palindrome Routine Part 2 for the Camp Talent Show. Part 2 of the routine directly followed Part 1 and

also utilized palindromes. Part 2 formed a palindrome that visually worked inward from the outer edges of the arc, in towards the center, and then back outwards. Thus, section one and five began, then working inwards toward the center of the arc, the classmates from section two and four clapped next, and then group three. Then, sections two and four performed again, and finally sections one and five finished the routine. Also, each of the three individual portions of Part 2 formed palindromes within themselves.

Day 8

On the eighth day, the students began class with a mirroring icebreaker that corresponded with the concepts of the main activities that day. After the icebreaker, two volunteer violinists from the class sight-read the Mozart Mirror duet. First, the two violinists played their parts one at a time, and then together, facing each other from opposite sides of a single flat (like a table top) stand.

The next activity of the day was the Bach Crab Canon Möbius Strip Activity. The class was shown a short video clip on Bach's crab canon that gave a visual aid through the use of a Möbius Strip (<https://www.youtube.com/watch?v=xUHQ2ybTejU>). Then, the class was shown a video on a how to create a Möbius Strip (<https://www.youtube.com/watch?v=BVsIAa2XNKc>). The students created their own Möbius Strips out of staff paper, composing music on both sides of the strip and the twisting and connecting the strip, just as the video instructed. Finally, some students went to the piano and performed their Möbius Strips for the class.

Day 9

On the ninth day, students watched a short video clip on musical fractals (<https://www.youtube.com/watch?v=V5tUM5aLHPA>). This video demonstrated an application of fractals in music with the example of Bach's Contrapunctus VII. After watching the video, the students formed six groups and created their own musical fractal with pipe cleaners and cotton balls. They first composed a single voice and then manipulated that motive through augmentation, diminution, or inversion. Finally, they performed their musical fractal for the class.

Day 10

On the final day, class started with a performance of the third movement of Haydn's *Symphony No. 47*, given by the counselors because this movement is a palindrome. The students were asked if they could guess what technique this piece used, and then they were shown that it is indeed a palindrome. Then a Powerpoint was shown providing other examples of musical palindromes that had not already been discussed. Such examples included *Lytic Suite* by Berg, *Ludus Tonalis* by Hindemith, and *Symphony Op. 21* by Webern.

Next, the significance of the numbers that the students provided on the first day, which were hanging on the butcher paper at the front of the room, were finally revealed. The students looked at the numbers first for palindromes. Then, a Finale file was played for the class, and they noticed that the piece in Finale was compiling each of the student's numbers in pitch form, following the standard compositional key (C=0, C#=1, D=2...etc.). Each measure represented a student. Then the students got themselves in line in order of the sequence of their measure in the piece. The

score of the piece was on the screen at the front of the room for the students to reference and to help them find their place in the order.

Then, an example of an unstructured piece was played for the class, to act as a contrast or a control, in comparison to the various structure techniques that we had analyzed throughout the class. The unstructured piece was titled, *El Iba a Otro Mundo* by Gorgina Derbez.

Finally, the students were given the “Final Survey” to complete in which they evaluated all the symmetries and forms of structures that had been discussed over the course of the ten classes (see Appendix for “Final Survey”). Thus, they determined which structure they found to be the most effective.

Class Results

Day 1- Listening and Drawing Activity

The paintings from the Listening and Drawing Activity, in which Bartok’s *Music for Strings and Percussion*, and *Celesta* and the *Faith of our Fathers* hymn were played, were separated by piece and by whether they illustrated symmetry. The top image, Figure 15a, shows the left wall of the classroom on which all of the asymmetrical paintings were hung, clearly divided with all the asymmetrical Bartok (Piece 1) paintings on the left and the asymmetrical hymn (Piece 2) paintings on the right. Figure 15b displays the right side of the classroom in which all the symmetrical paintings from Piece 1 and Piece 2 were hung. The symmetrical

paintings from the Bartok (Piece 1) listening were on the right, and the symmetrical paintings from the hymn (Piece 2) were on the left, thus the Bartok were hung closest to the front board on either wall, and the hymns were hung closest to the back door.

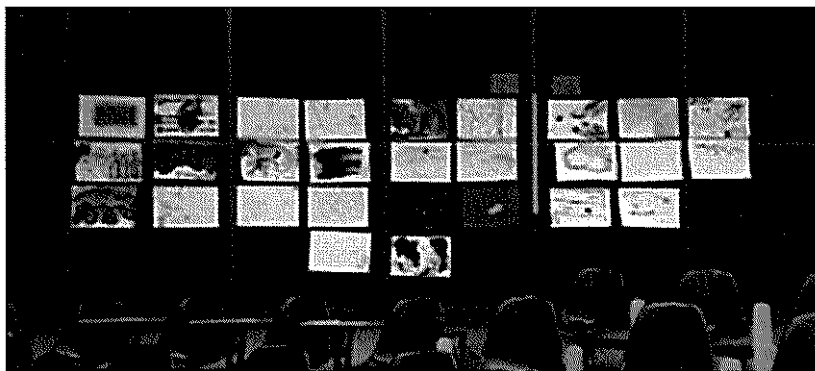


Figure 15a- Asymmetrical Paintings

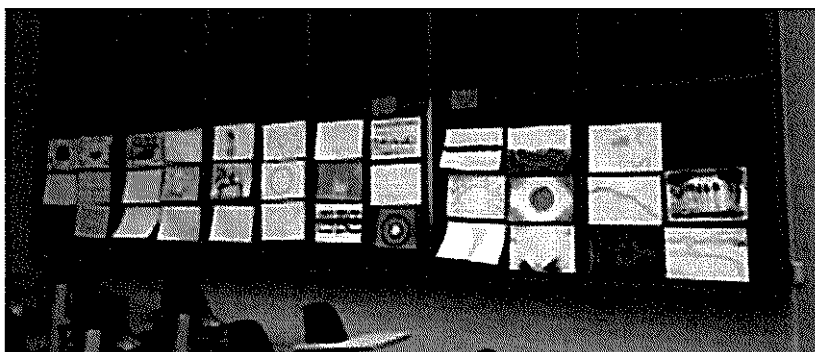
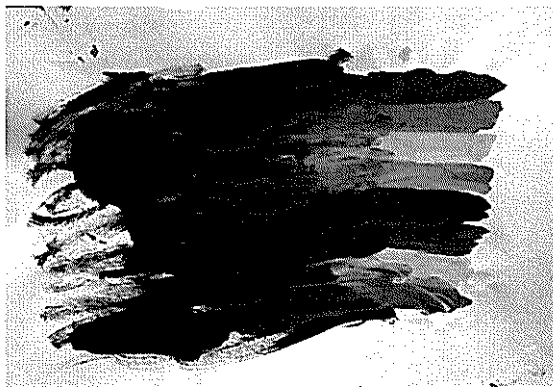


Figure 15b- Symmetrical Paintings

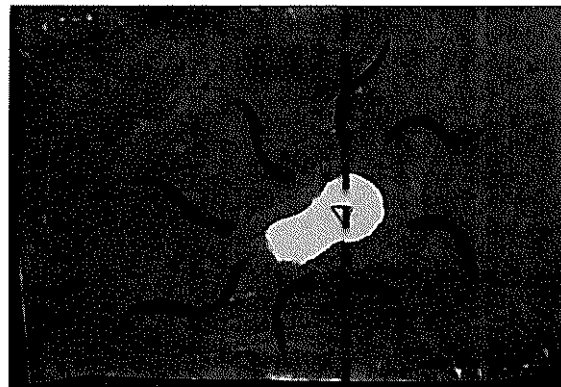
When the paintings were separated like this, it was easy to see a trend in the number of asymmetric paintings for the Bartok listening and the number of symmetric paintings for the hymn. Figure 15a presents more than twice as many asymmetrical paintings on the left side, meaning paintings drawn during the Bartok listening than those drawn during the hymn, thus conveying the asymmetrical nature of the Bartok. Similarly, Figure 15b shows over double the amount of symmetrical paintings on the left, hence for the hymn. This is noteworthy because the hymn is very symmetrical in nature in that it is repetitive and has a very

structured shape. The students were not given any information on the pieces before they painted, yet they intuitively painted the shape of the piece respective to the overall form, whether it was asymmetrical or symmetrical.

Some specific examples of the students paintings that are especially intriguing are those presented in Figure 16. These paintings are all selections from the Bartok paintings, and they are particularly good examples of paintings that feature the golden section. In Figure 16a, the golden section occurs where the black fades into the pure color, in Figure 16b, the small red triangle within the yellow blob lies precisely at the golden section, 0.618 of the length of the paper, in Figure 16c the tallest green branch coincides with the golden section, and in Figure 16d, the inverse golden section ($1/0.618$) intersections directly through the center of the pupil. The students had no previous knowledge about the golden section or that Bartok's piece utilized this technique, yet they subconsciously were attracted to this appealing ratio.



a.)



b.)

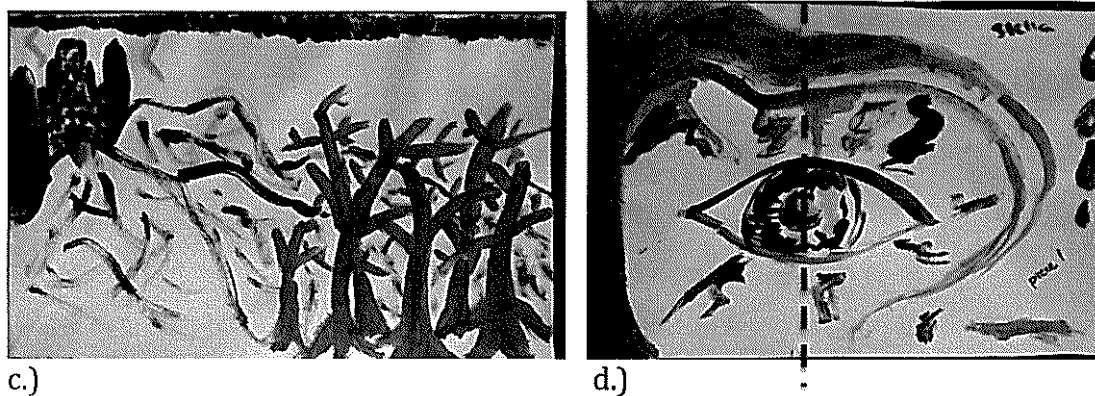


Figure 16

Additionally, certain paintings were selected from the hymn paintings that exemplified the nature of symmetry that the piece embodied. For instance, Figure 17a displays repetition of rectangles in various proportions and shows the alternating colors of purple and green just as the hymn has the same repeating shape throughout and simple alternates between the verse and the refrain. Figure 17b, 17c, and 17d portray symmetry in the form of a bulls-eye. Any way the image is dissected, there is an axis of symmetry. Also, the attention is drawn toward the center of the image in these paintings, opposed to 61.8% of the way through the page like the paintings from the Bartok in Figure 16. Again, the students were not informed of the structure of the piece before the listening activity. The students demonstrated the asymmetry of the Bartok and the symmetry of the hymn in their paintings, whether they were aware of it, or not.

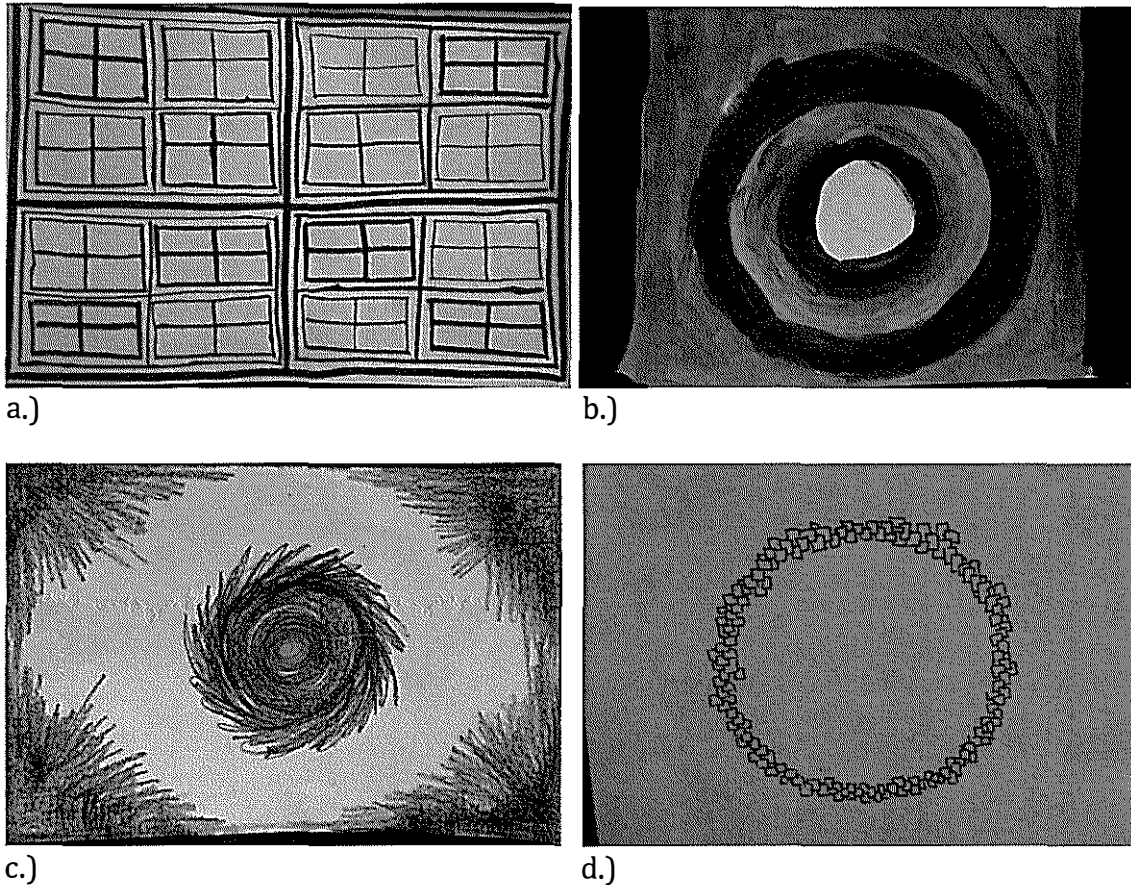


Figure 17

Day 2- Line Test

After the Line Test was administered, the students' individual slips of paper with their hash line were calculated in terms of the ratio of the 16cm line to the length up to their hash line, and then the ratios were averaged together. As a result, seventeen of the thirty students ($17/30 = 0.57 = 57\%$) placed their mark within the golden section range or the inverse golden section range. The golden section range encompassed the 8-12cm portion of the line, which is (0.50- 0.75) and the inverse golden section included the range between 4-8cm (0.25- 0.50). Four of the students

placed their mark exactly at the 0.618 location; however, these ranges were used because the golden ratio is a general trend and allows a small margin for error, hence it is not always exact. Also, the inverse golden section was included in this 57% class average because the inverse golden section is just as appealing as the true golden section, thus it is just inverted. It is simply a matter of the image being processed left to right, or right to left. It is also interesting to note that of the thirty slips received, two students placed their mark on the extreme edge of the line (at either 0% or 100%), three students placed their line directly in the middle thus symmetrically, and twenty-five placed their line asymmetrically on the paper. There was an overwhelmingly obvious trend of students being drawn to an asymmetrical location on the line as opposed to a symmetrical placement. All data were collected from the individual slips that the students turned in, but the poster in Figure 18 serves as a comparative visual aid to convey where students were drawn to on the line.

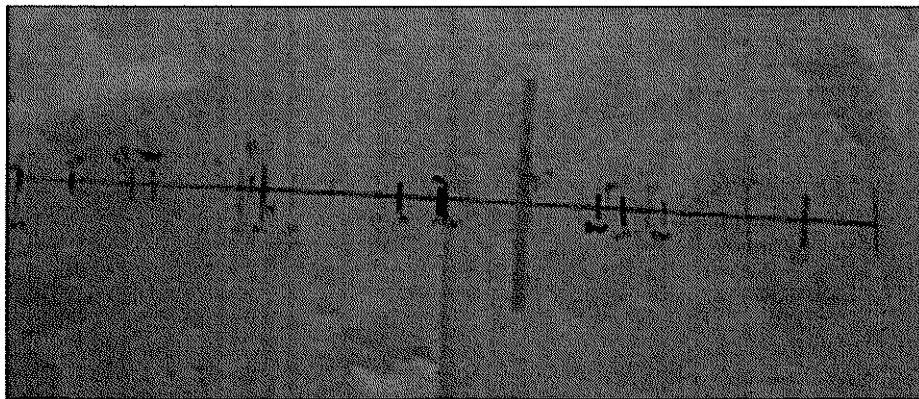


Figure 18

Day 3- Debussy Listening Survey

The Debussy listening, *Reflects dans l'eau*, represents music utilizing the golden mean, as this piece climaxes at the golden section. The students, unaware of the presence of the golden ratio in the music, gave this piece an average of 7.61 on a scale of 1-10, with 0 standing for least pleasing and 10 for most pleasing. The data convey a mode of 7 and a clear favoring for the ratings above 5. Not a single student chose the ratings between 0-5, but rather all 28 participants chose a ranking between 5-10.

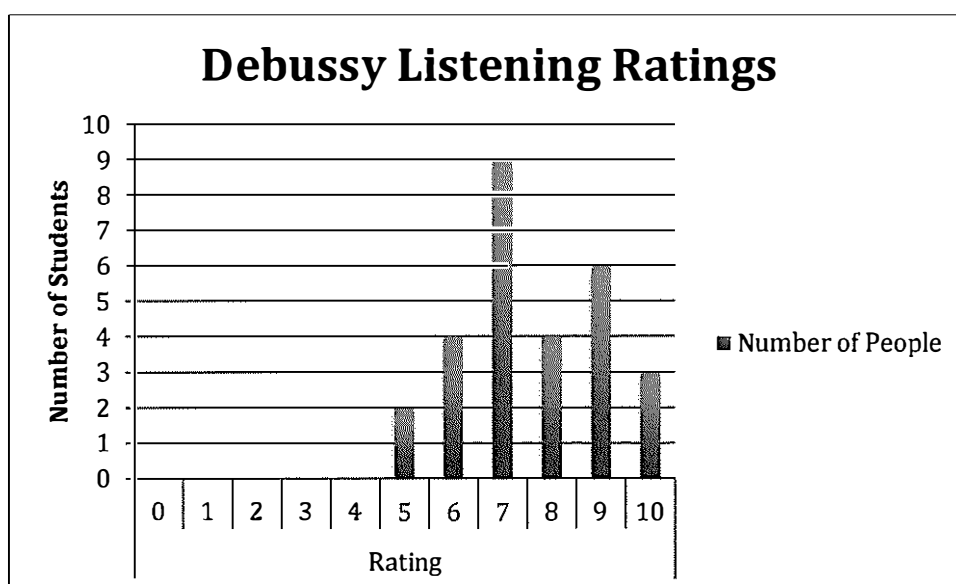


Figure 19

In addition to the overall rating of the piece, the survey asked the students to provide colors, shapes, and imagery that came to mind while listening to this piece. The choice of colors from the class was heavily weighted on the color blue. Students even specified light blue or dark blue on some occasions. While four students described the piece as purple and three students described it as green, fourteen chose blue, three chose light blue, and three chose dark blue. Thus, a total of twenty students chose the color blue to represent *Reflects dans l'eau*. This makes sense

because the title means reflections on the water, and obviously the students caught on to the thematic material that Debussy used to create a “blue” sound. Other imagery or descriptions that students included were “calm,” “peacefulness,” “sky and clouds,” “outdoors,” “water falls,” “waves,” “night,” and “serenity.” Students also mentioned their satisfaction with the “ending with a large climax” and “a clear and resonant resolution.”

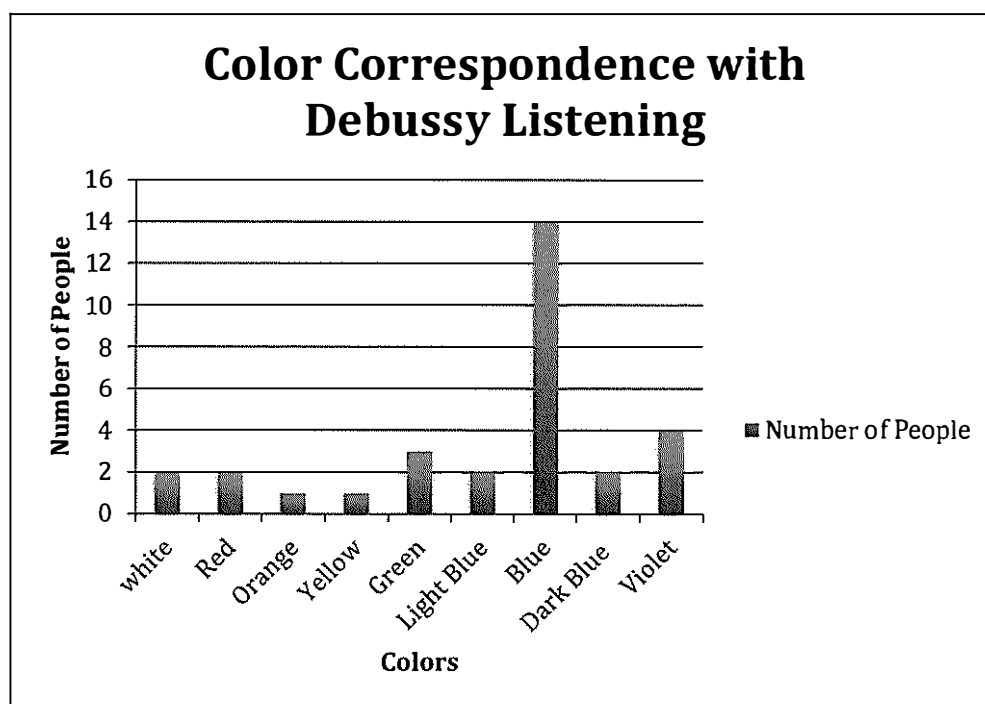


Figure 20

These comments are significant because Debussy placed the major climax at the golden section, which is supposedly the most appealing location.

There is much consistency in the students’ choice of colors and descriptions for this piece, showing that Debussy portrays a specific image very clearly, and in the case of this study, elicits the listeners’ general approval.

Day 5- Fun with Fibonacci
Da Vinci Golden Section Body Proportion Exercise

NAME	B/N	F/K	L/H	A/E	X/Y
Miranda	1.658	1.75	1.95	1.706	2.667
Teresa	1.737	1.5	1.842	1.75	2.667
Dominic	1.67	1.5	1.82	1.53	3
Nathaniel	1.63	1.64	1.95	1.72	2.75
Julia	1.65	1.7	2	2.23	1.75
Kira	1.64	1.7	2.1	2.45	1.62
Autumn	1.625	2	2	1.63	3.2
Abby	1.53	1.73	1.74	1.74	2.3
Stella	1.68	2.3	2	1.7	2.6
Caroline	1.68	1.5	2	1.81	3
Michael	1.64	2	2.08	1.65	3.6
Sam	1.72	1.75	1.95	1.62	3.4
Sarah	1.65	2	2	1.68	2.375
James	1.64	3.08	1.82	1.64	2
David	1.62	1.94	2.16	1.79	3.4
Patrick	1.33	1.75	2.24	1.75	3.8
Christina	1.615	2	1.846	1.703	3.2
Lexi	1.625	2	1.921	1.813	2.538
Tiffany	1.626	1.68	1.94	1.687	3.9
Paige	1.6282	1.756	1.9113	1.809	4.4
Emma	1.56	1.5	1.95	1.59	1.6
Ella	1.58	1.33	1.85	1.5	2
Ellie	1.56	1.5	1.68	1.69	1.6
Amelia	1.67	1.36	1.73	2	2.43
Laura	1.58	1.71	1.95	1.53	3.1
Madison	1.6	1.71	1.9	1.63	3.8

Figure 21a

The chart in Figure 21a displays the ratios of body measurements for all the students in the class. The ratios that were being measured here were the student's height and navel height (B/N), the length of index finger and the distance from fingertip to first knuckle (F/K), the length of leg and the distance between hip and

kneecap (L/H), length of arm and the distance from fingertip to elbow (A/E), and the length from top of head to bottom of chin and the length of the bottom of the ear to the bottom of the chin (X/Y).

The ratios of B/N averaged to 1.62, the ratios for F/K averaged to 1.78, the ratios of L/H come to an average of 1.94, the ratios of A/E have an average of 1.74, and the ratios of X/Y averaged to 2.80. From these numbers, it is seen that the ratio of the students' height and their naval height yielded results closest to the golden proportion, 1.618. All of the ratios for (F/K), (L/H), (A/E) were within the general golden range; however, the top of head to bottom of chin and the length of the bottom of the ear to the bottom of the chin (X/Y) were quite far from the 1.618 golden ratio. The ratios for (X/Y) averaged to 2.80, and there are some particularly high ratios in this category, such as 3.8, 3.9, and even 4.4. From the spread of data, it can be deduced that this is not merely miscalculation since it seems to be consistent among the students. The ratio of (X/Y) is not as accurate as the other four ratios, and it is more of a stretch to claim that it embodies the golden ratio. Figure 21b illustrates a comparison of the different ratios. It is clear that most of the ratios oscillate around the 1.618 mark, but the (X/Y) ratio soars above the others producing a much higher average than 1.618.

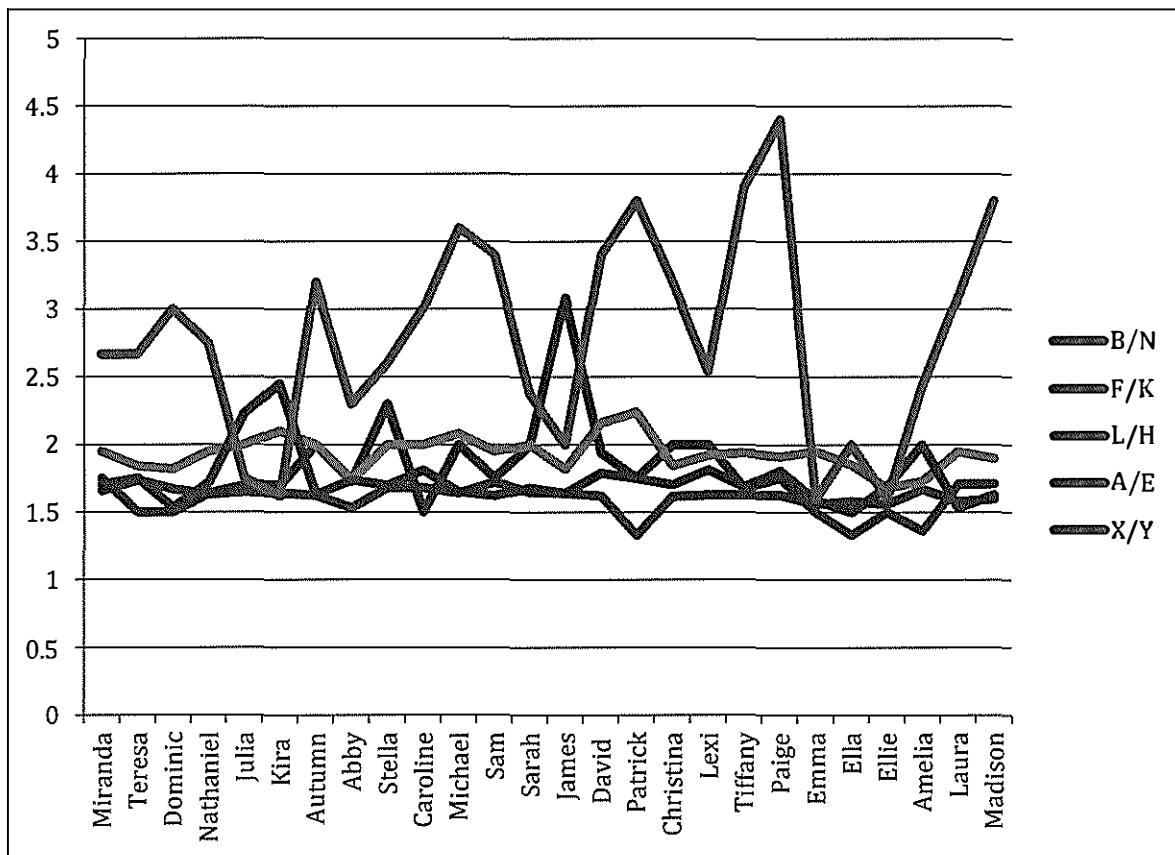


Figure 21b

Day 6- Machaut Listening Survey

The Machaut listening, *Ma fin est mon commencement*, demonstrates the musical palindrome. Following the same procedure as the Debussy listening on Day 2 the students were unaware of the title of the piece and of the structure of the piece as they completed the survey. The average rating for the Machaut listening survey was 3.58 and the most commonly selected rating was 2. Observing the results in the graph shown in Figure 22, it is evident that the ratings are much more spread than in the Debussy listening results. Whereas the Debussy ratings were confined to the 5 to 10 range, the Machaut ratings, contrastingly, span from ratings of 0 to 9, and the

number of people selecting each rating was more consistent. The average, 3.58, is also much lower than the average rating of 7.61 for the Debussy survey.

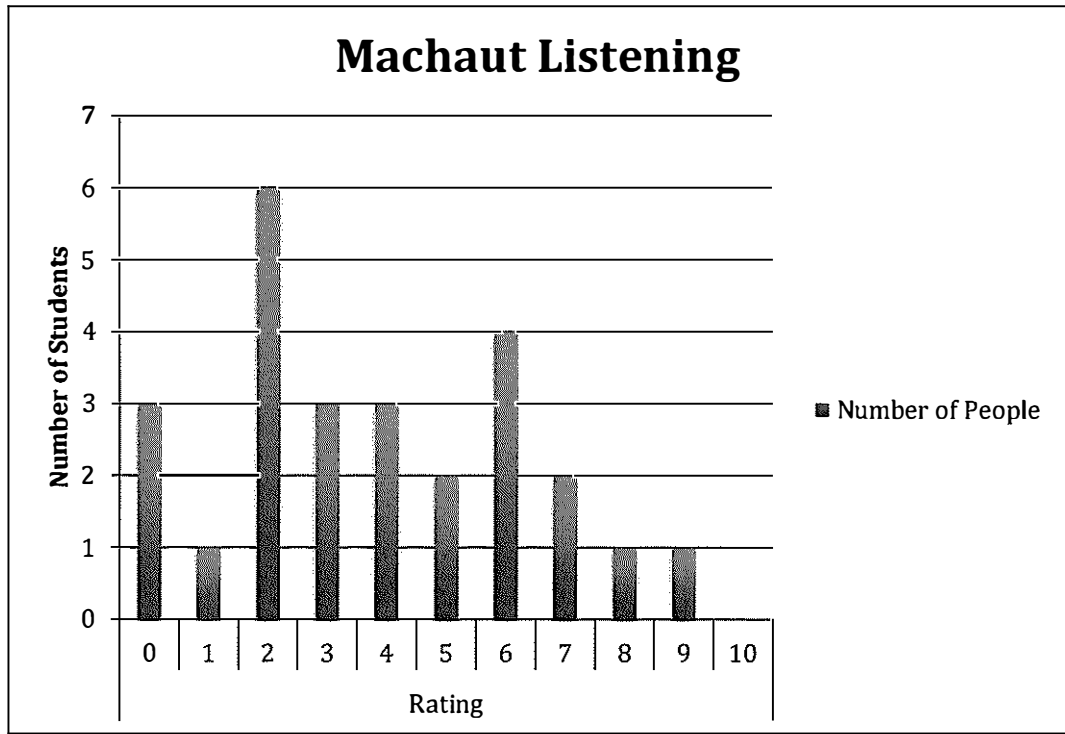


Figure 22

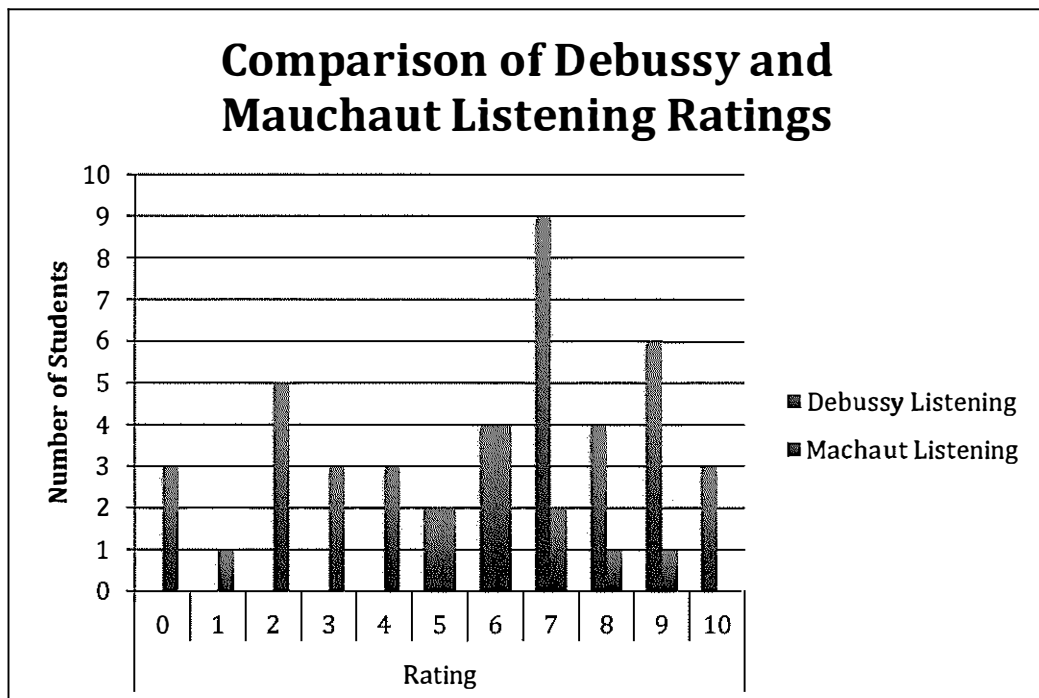


Figure 23

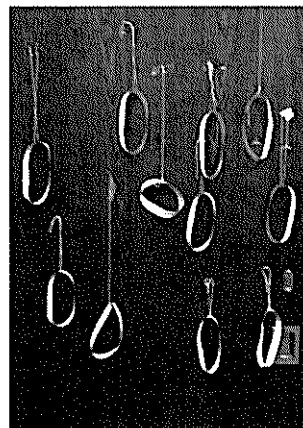
Figure 23 compares the ratings of the two pieces side by side so that it can clearly be seen that Machaut had a larger range of ratings while the Debussy had generally higher ratings.

Day 8- Crab Canons and Möbius Strips

The student's Möbius strips offer a visual aid to conceptualize the intriguing crab canon. The compositions that the students wrote on their staff paper Möbius strips resulted in an infinite piece that alternated between one side of the slip of paper and the other side. The twist of the paper, followed by the connecting of the two ends of the paper, caused the front and back to fuse together the interweaving. Figure 24 a and b show some samples of the students' Möbius strips, and in Figure 24a, the close up view allows the transition of sides and clefs to be visible.



a.



b.)

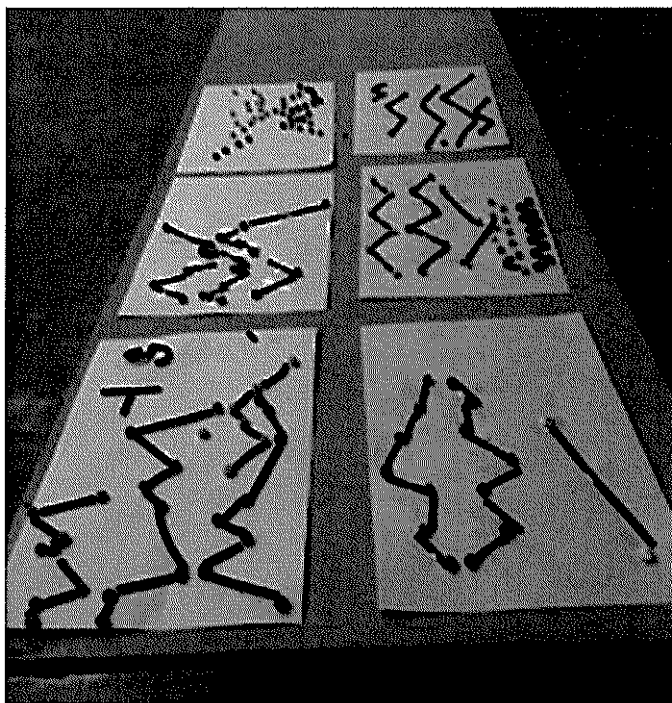


c.)

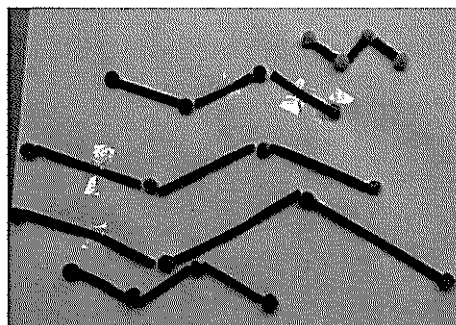
Figure 24

Day 9- Musical Fractals

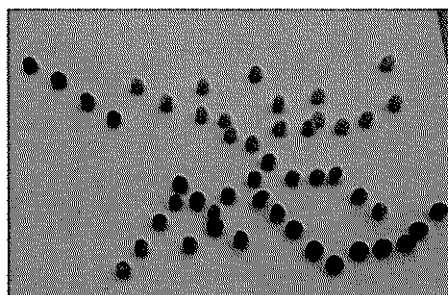
The music fractals created by the students demonstrate the techniques of augmentation, diminution, and inversion as they first composed a motive and then altered that original motive. Their creative liberties can be observed as all six groups came up with different results (see Figure 251a). A particularly unique example is found in Figure 25c in which the group decided not to use pipe cleaners, which represented the durations or lengths of the notes. Instead, this group still spaced out their pitches, or cotton balls, so that it possessed a rhythm, but the lack of pipe cleaners conveyed the intention of a staccato style. When that group performed their fractal for the class, they did not sustain the notes, but rather separated all of them.



a.)



b.)



c.)

Figure 25

Day 10- Culmination Day

The results from the last day summarized the effectiveness of the various mathematical patterns discussed throughout the class. The Final Survey asked

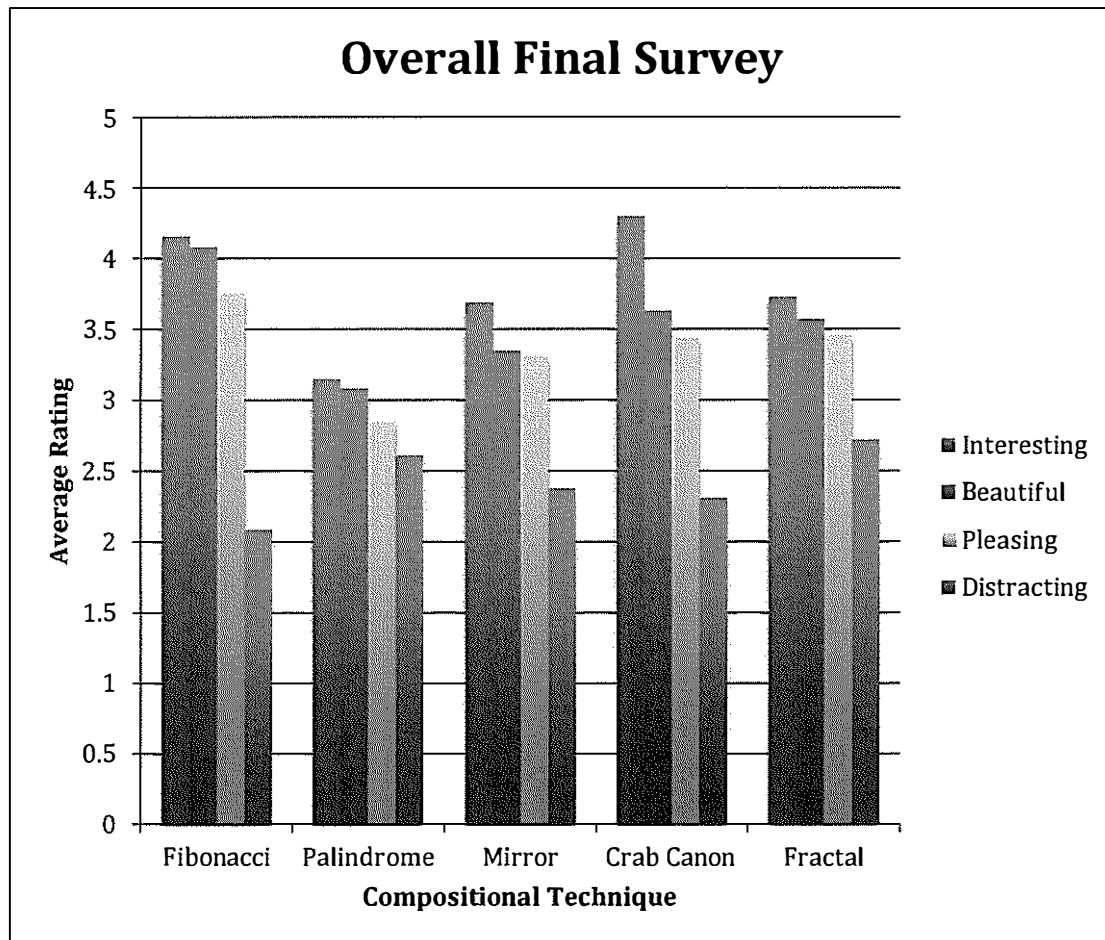


Figure 26

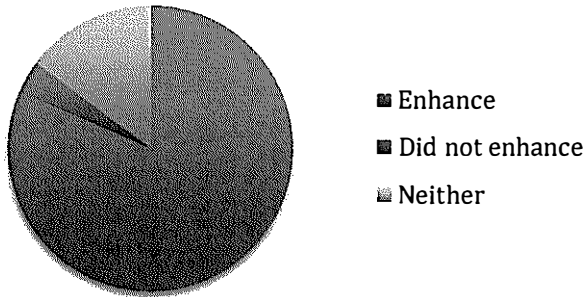
the students to rate each of the compositional techniques based on the following criteria; provoking interest, possessing beauty, overall pleasing, or overall distracting. Figure 26 displays a comparison of the averaged results to the survey. Within all of the compositional techniques, the level of interest that each technique

provided yielded the highest averages, as the tallest bars in the graph signify 'interesting.' Analyzing the different techniques side by side, the crab canon had the highest average rating for level of interest, with an average of 4.30 on a scale of 1 (poor) to 5 (best). The Fibonacci numbers had the highest average rating, 4.08, for possessing beauty as well as the highest average rating, 3.73, for sounding overall pleasing. This is particularly interesting because many sources claim that the Fibonacci numbers and the golden ratio embody aesthetic beauty and are naturally pleasing. These averages from the students appear to support those claims. Lastly, the fractals had the highest average rating of 2.72 for causing a distraction. In class, students discussed how some techniques worked to enhance the music while other techniques may have gotten in the way, leading it to cause a distraction to the listener. In the Final Survey, students were asked to determine if each of the techniques generally enhanced or made the pieces that utilized them more appealing. The pie charts in Figure 27 illustrate the results to this question. From the charts, it can be seen that the Fibonacci numbers and golden mean had the most extreme results of all the mathematical techniques, as it had the largest majority of students decide that the techniques enhanced the compositions. Specifically, 22 of the 27 students chose "enhance." The palindrome was the only technique for which more students did not select "enhance." Also, the palindrome pie chart (Figure 27b) appears to be the most balanced, in that it is very close to a three-way split among the choices; enhance, did not enhance, or neither. The mirror, crab canons, and fractals yield almost exactly the same results from this final survey. In all three of these categories 14-16 students determined that these techniques enhanced the

pieces, 8-9 students determined that they did not enhance piece, and 3 students chose neither.

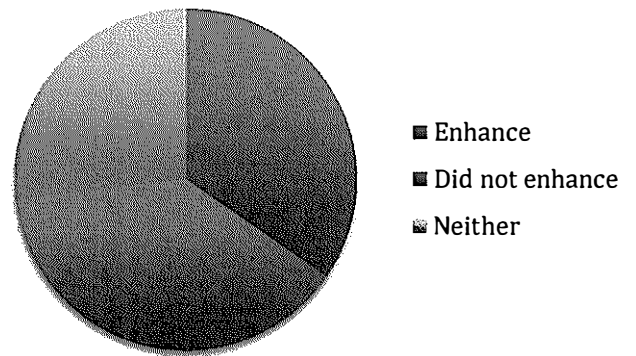
The last question that was presented in the Final Survey was, “Which technique was most effective?” As a result of this question, the Fibonacci numbers was chosen most frequently as the most effective technique. Further, 11 students chose the Fibonacci technique as the most effective, crab canons had the second highest total with a count of 8 students, fractals had a total of five students, palindromes had a total of 2 students, and mirrors was chosen by 1 student to be the most effective (see Figure 28). Clearly, the results tend to show that the use of Fibonacci numbers or the golden ratio in music is the most effective, as well as the most interesting and most pleasing mathematical technique.

Fibonacci/Golden Mean



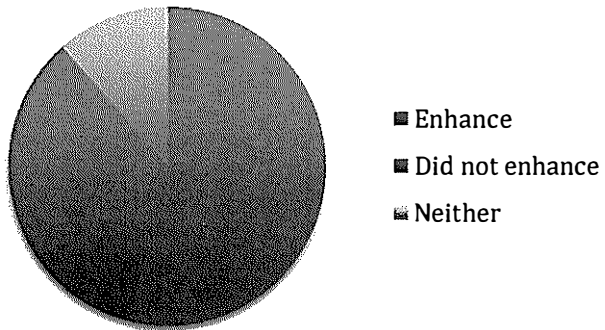
a.)

Palindrome



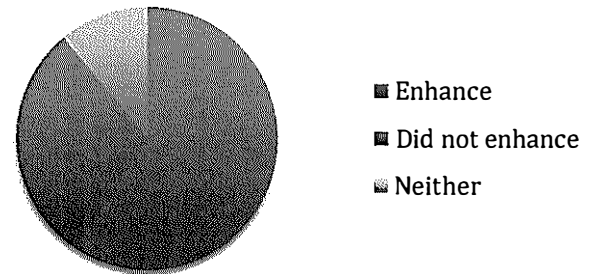
b.)

Mirror



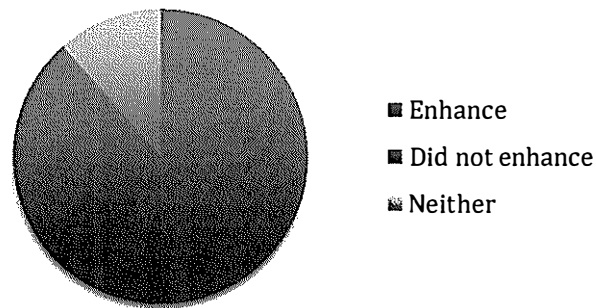
c.)

Crab Canon/Möbius Strip



d.)

Fractal



e.)

Figure 27

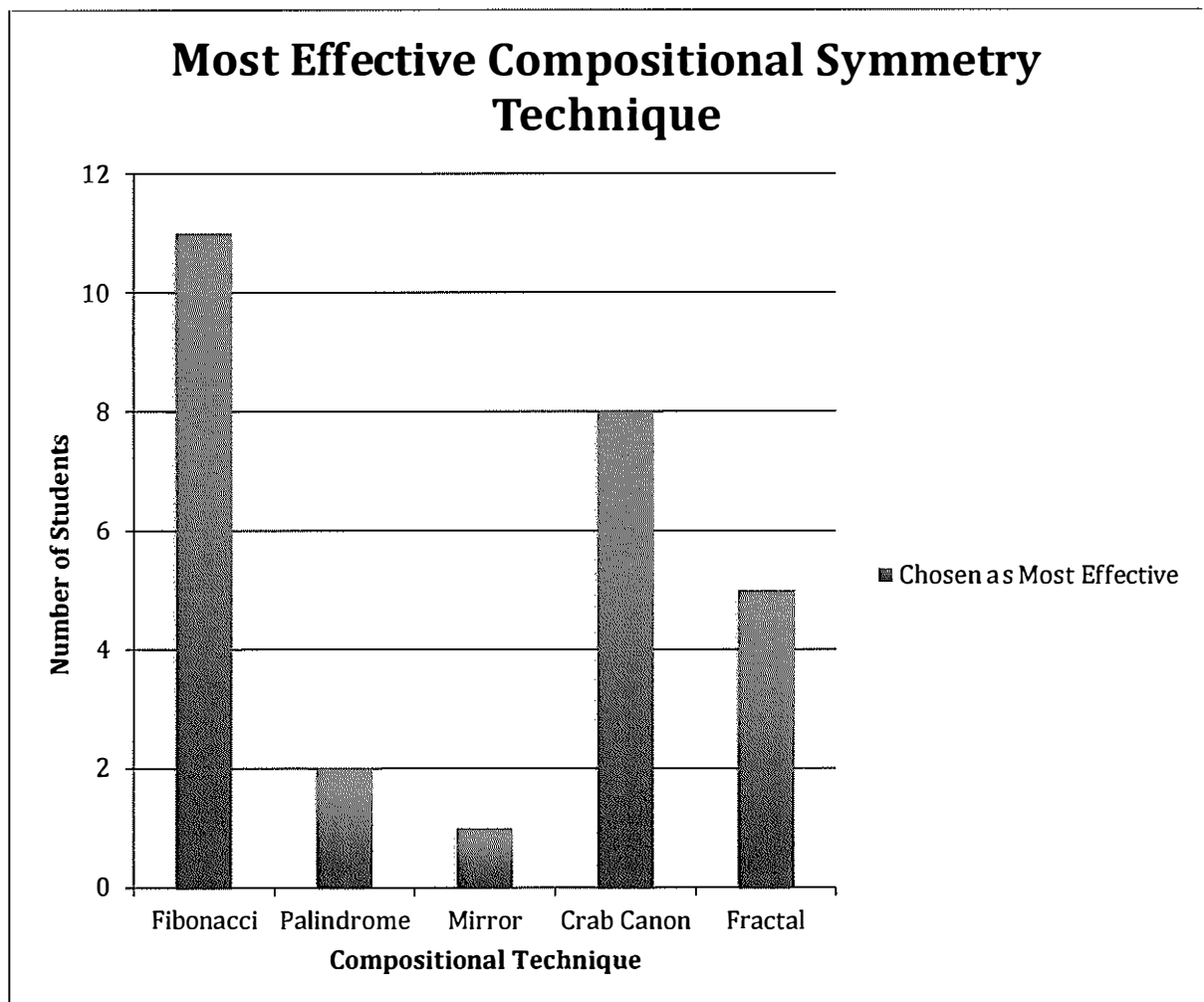


Figure 28

Conclusion

This study investigated mathematical patterns such as the Golden Mean as it applies to the composition of classical music, in comparison to other mathematical symmetries used as compositional tools, such as palindromes, crab canons, and fractals. This research explored the impact that these compositional techniques have on the style and structure of a composition as a whole, and thus considered how these techniques set the piece apart from other works that did not use such mathematics.

From the results gathered of all the asymmetrical and symmetrical techniques being analyzed, the Fibonacci numbers and the golden section appeared to be the most effective compositional tool utilized. Previous research and exploration done prior to the experimenting indicated that the Fibonacci series and the golden ratio exemplified natural beauty and pleasure, and the results from this experiment provided support for that claim. The asymmetrical proportion offers a desirable trait that the symmetrical techniques such as palindromes, crab canons, and fractals do not possess. Additionally, exploring varieties of symmetrical techniques revealed interesting discoveries. The palindrome, possibly the most symmetrical of all because it epitomizes identical consistency, was given the lowest ratings in terms of interest, beauty, and being pleasing overall according to the Final Survey results. In some cases, the amount of mathematical detail in a piece led to a distraction from the music instead of to an enhancement. This is not to say that all symmetrical tools are ineffective as musical compositional devices, since crab

canons yielded the second highest ratings as most effective compositional technique overall. Crab canons also depend on symmetry as the foundation of their structures.

However, the sample size in this study is relatively small, consisting of only about 30 students. Different results might have been attained if a larger population had been included. With more feedback acquired, more representative and possibly more credibly sound data might be gathered.

This study demonstrated a method of exploration in which a field as subjective as music can be analyzed objectively. This study changes the typical perception of music into a logical mathematical perspective. Thus, the transformation from pitches and rhythms to colors, shapes, images, and numbers, generates an objective field. By embracing this unique perspective, the powerful connections between mathematics and music can be unleashed.

Appendix

Illinois Wesleyan Chamber Festival 2014

Unleashing Music's Hidden Blueprint Elective Class Research Consent Form

Dear Parent/Guardian,

My name is Natalie Hoijer and I am a senior at Illinois Wesleyan University studying music and mathematics. I have been a counselor and have taught an elective course at the Chamber camp for the past two years and this year I am leading a class on hidden symmetrical structures that are embedded in music. I am putting together a research project on this topic and I am planning on giving surveys to the class and using anonymous data from the class as part of my research project. No names of students will be included in the research. The surveys are just a way for me to gather so feedback from young musicians.

I give my (son/daughter) _____ permission to
_____ participate in anonymous surveys that will take place in the camp elective class this year, *Unleashing Music's Hidden Blueprint*.

_____ include portions of the student's work as an example of the studies in the camp elective class this year, *Unleashing Music's Hidden Blueprint*.

_____ be photographed in group activities during the camp elective class this year, *Unleashing Music's Hidden Blueprint*. Photos may be published for my research studies at Illinois Wesleyan University.

Again, all data and comments from the class included in the research will be strictly anonymous and no names will be included in the project or in conjunction with photos.

Parent Name _____

Parent Signature _____ Date _____

Student Name _____

Student Signature _____ Date _____

Thank you so much for your participation!

Natalie Hoijer

Camp Counselor
Elective Teacher
Illinois Wesleyan Student

IWU Contacts:

Mario Pelusi: mpelusi@iwu.edu

Lisa Nelson: lnelson@iwu.edu

Brian Brennan: brennan@iwu.edu

Lesson Plans

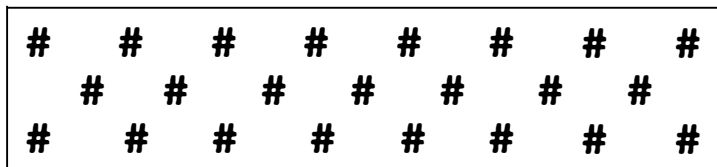
Day 1

- Introduction

-Myself & Class

- Class Introduction Poster

-Have students say their name and a number (100-9,999) that they connect with or that represents them. Write the student's names and numbers on a large piece of butcher paper and hang in the room for a future activity.



-**Activity:** Listening and Drawing Activity

-Have students listen two pieces of music, one utilizing the golden section and one without. First listen, just absorb the music. While the first piece is playing a second time, have them draw/paint what they hear (shapes, colors, imagery) and creatively use the whole sheet of paper. Then have the students do the same thing for the second piece.

Pieces: Bartok (golden section) vs. Hymn (verse refrain)

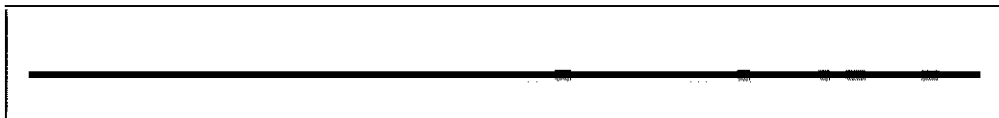
(Music for Strings, Percussion and Celesta vs. Faith of our fathers)

Day 2

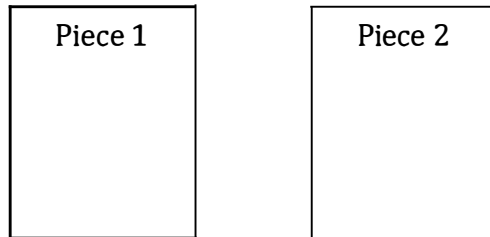
-**Activity:** Paper with line on it

-Give students a slip of paper with a single line and have them place a mark on the paper not knowing anything about the golden section.

-Then have them place their marking on a larger line on butcher paper at the front board to compare results



- Have each student share their paintings with the class
 - As each student presents their interpretations of Piece 1 and Piece 2, take notes on butcher paper posters of reoccurring comments regarding Piece 1 and Piece 2.



Day 3 –Debriefing and Explaining Golden Section

(Before class, hang up students paintings on the classroom walls. First divide them by Piece 1 and Piece 2, then further divide them into symmetrical Piece 1 paintings, asymmetrical Piece 1 paintings, symmetrical Piece 2 paintings and asymmetrical Piece 2 paintings. Hang all the symmetrical on one wall and all the asymmetrical on the opposite wall. Label Piece 1 and Piece2, but do not label symmetrical and asymmetrical.)

Piece 1
Asymmetrical

Piece 2
Asymmetrical

Piece 2
Symmetrical

Piece 1
Symmetrical

- Debussy: (Play recording of Reflects dans l'eau)
 - Listen to piece and have students complete survey
 - On the back of the survey, ask students to study the paintings and make observations as to how the paintings are grouped. Ask if they know why they are separated the way that they are.

- **Activity:** Golden Section Painting Survey
 - Have students take the survey and choose one of the two paintings from each category that seems most appealing
 - Discuss results from the painting survey
 - Show Powerpoint of the Paintings and have the students stand at the left or right side of the classroom, depending on which painting they thought to be more pleasing
- Discuss Results from Line Test
- Discuss results from student's paintings
 - (Before Class: Hang up the student's paintings, separating those that are symmetrical and those that are asymmetrical)
 - Have students make observations on their paintings
- Discuss Debussy Survey Results

Day 4

- Ask class if anyone encountered Golden Ratio in the past 24 hours
 - Make a list
 - Share examples
- Video Clip on Fibonacci: <https://www.youtube.com/watch?v=0tAZe6pP-FM>
- Advertising with Fibonacci
- Activity:** Mozart Sonata Activity
 - Divide students into groups and give them each a Mozart Piano Sonata (279, 280, 283)
 - Have them calculate and search for the Golden Section
 - Student groups share findings with the class
- Take a look at the Golden Section in Modern Pop tunes
- Class discussion:
 - *Is the Golden Section intuitive or intentional?
 - *Is the Golden Section/climax appealing? Why? How?

Day 5– Fun with Fibonacci Day

-**Activity:** Da Vinci Golden Section Body Proportion Exercise

- Have students pair up and measure certain body proportions to see if they have a golden proportion
- Complete Handout
- Share as a class

-Song Lyric Game

Day 6

- **Activity:** Listening to a piece that uses Palindrome

(Machaut- Ma fin est mon commencement)

- Complete survey
- Discuss if piece is pleasing based on colors, shapes, imagery, emotions

-As a class, brainstorm palindromes

- **Activity:** Palindrome Icebreaker Activity

-Pass out index cards with a number in them and have students get in line in the form of a palindrome without talking (adaptation of birthday team building exercise)

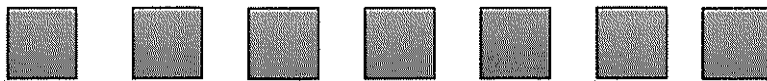
-Learn Palindrome Routine Part 1 for Camp Talent Show

Day 7

-Lulu-Berg

- **Activity:** Memory Card Game

-Divide students into groups of 5, give them a set of Lulu Memory Game index cards and have them find the pairs that match up. Then have them figure out the chronological puzzle and place the cards around the fermata, creating the palindrome. The first time, the class will be divided into three smaller groups, then once they master that, we will try one big palindrome with the whole class.



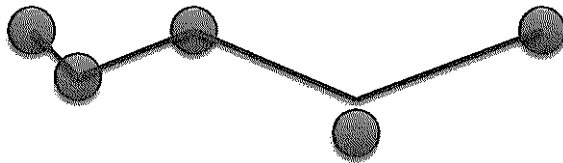
-Learn Palindrome Routine Part 2 for Camp Talent Show

Day 8

- Mirroring Ice Breaker
- Mozart Der Spiegel Duet (Mirror Duet)
 - Have two students perform the duet for the class (first one violinist at a time, and then together from opposite sides of the stand)
 - Discuss the mirror technique and the Golden Section used!
- Activity:** Bach Crab Canon
 - Show Crab Canon Video:
 - <https://www.youtube.com/watch?v=xUHQ2ybTejU>
 - Show Möbius Strip video:
 - <https://www.youtube.com/watch?v=BVslAa2XNKc>
 - Make Möbius Strips!
 - Have students compose music on both sides of a strip of staff paper, then twist the column into a Möbius strip, and then pair up with a partner and try to read each other's Möbius strips. Then they will perform the Möbius compositions for the class

Day 9 Fractals

- Finish Möbius Strip Presentations
- Watch Video Clip on Fractals – Bach Contrapunctus VII
- Activity:** Break into groups and give each group a motive



- Have the groups create musical fractals by manipulating the instrumentation and durations of the motive
- Have groups perform their fractals for the class

Day 10

-Performance of Haydn Palindrome Quintet

- Discuss the remainder of the Palindrome examples:

-Lyric Suite-Berg

-Discuss the palindrome that is used and map/diagram it out

-Ludus Tonalis- Hindemith

-Discuss the palindrome in Praeludium and Postludium

-Discuss structural palindrome with the interludes (...Baroque, Romantic, March, Romantic, Baroque...)

-Webern Op. 21

-Class Discussion:

*Compare large - scale palindrome to small - scale detailed palindromes.

*Does the palindrome enhance or take away from the music?

*Is there a point where the palindrome details become a distraction?

*How does the palindrome technique compare/contrast with the Fibonacci or golden section technique? Which is more pleasing or which is more effective?

-Numbers in Composition

- Play the Finale recording of the piece composed of the student's numbers from the first day introductions

-Have the class guess what the composition is after one listening

-Reveal the significance of students' numbers from first day introductions

-Have students get themselves in order by the sequence of their contribution to the Finale piece. Each measure represents a student.

-Listen to Unstructured Piece (*El Iba a Otro Mundo* by Gorgina Derbez) and Discuss

-Take Final Survey

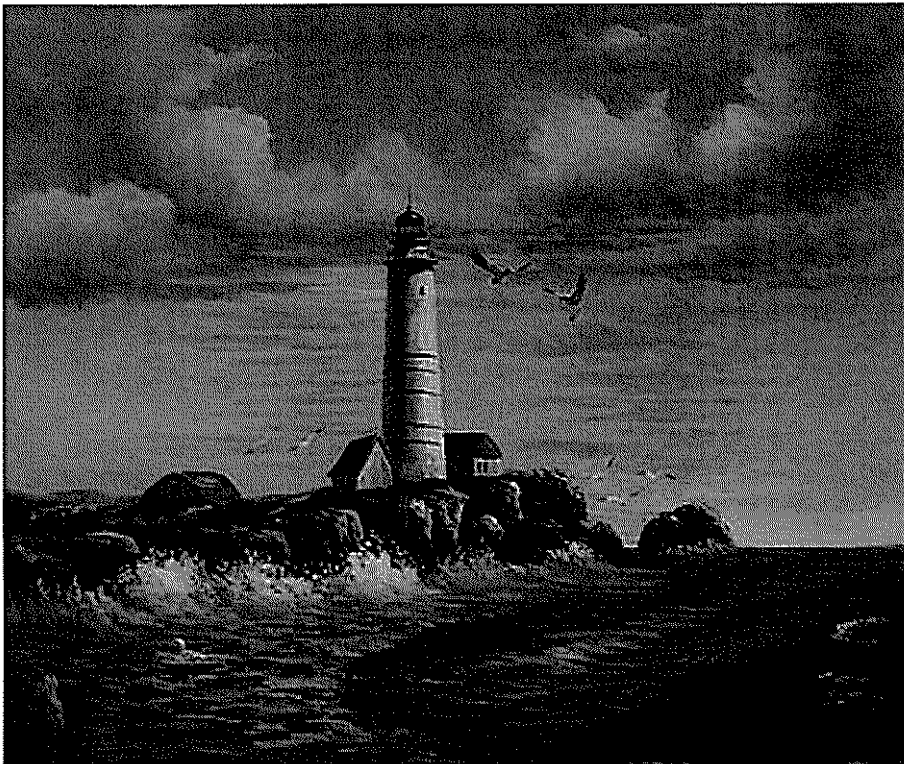
-Debrief all of the symmetries that we learned in the class

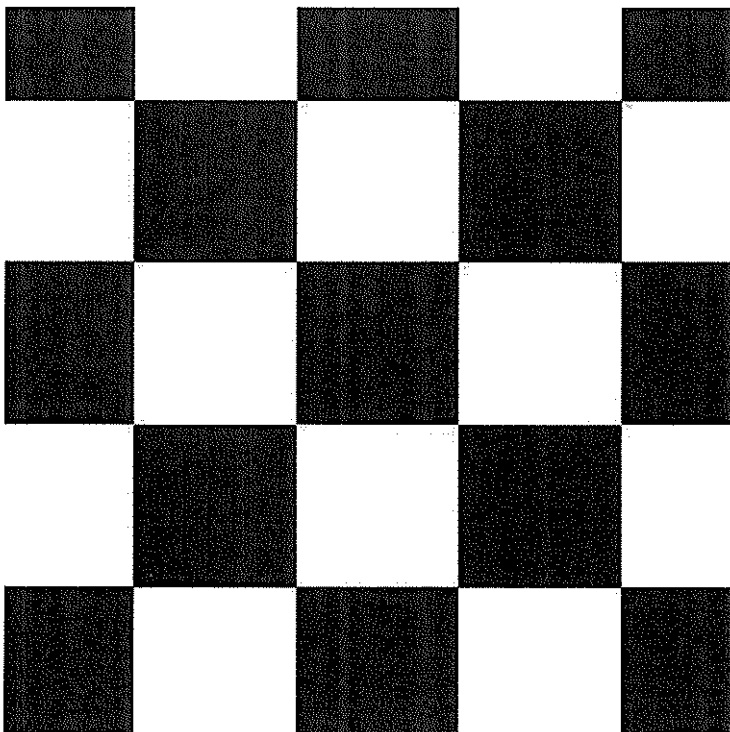
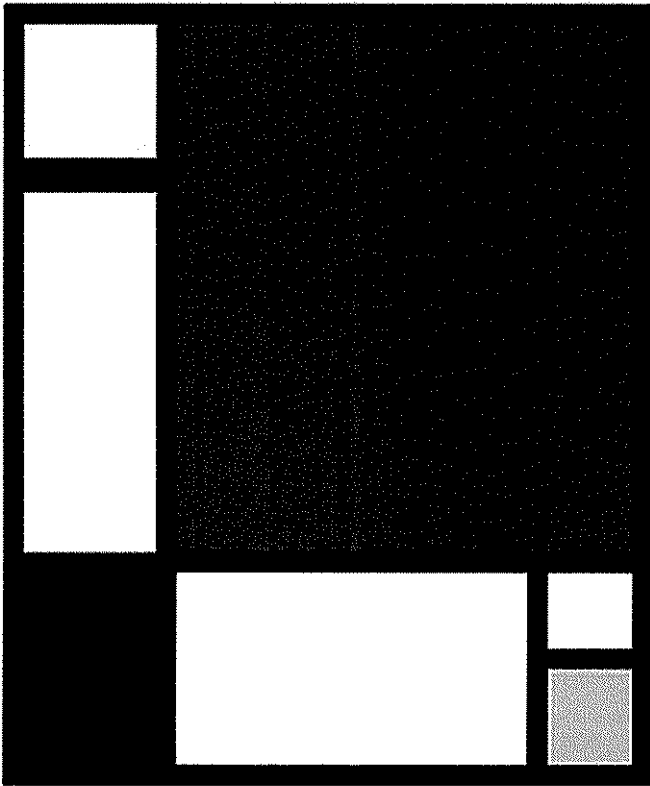
-Discussion: Compare

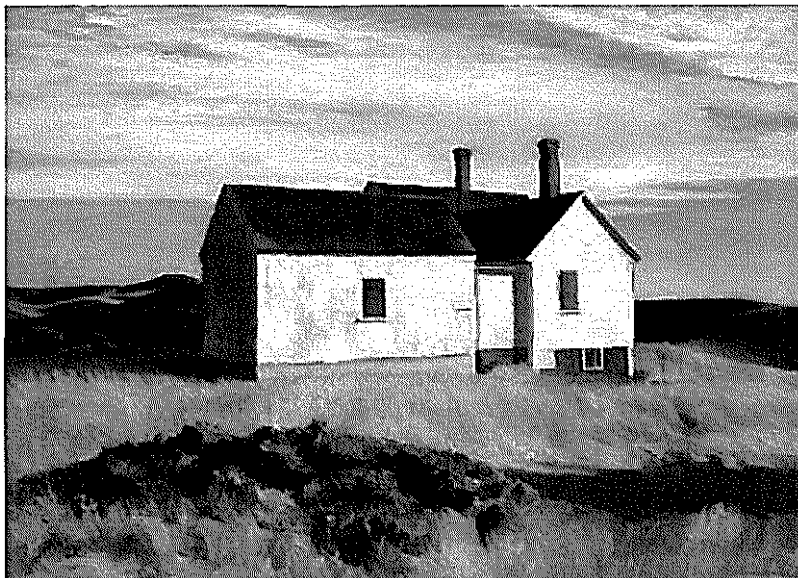
GS/Fibonacci vs Palindrome vs Crab Canon vs Fractal

*Which is more effective? WHY?

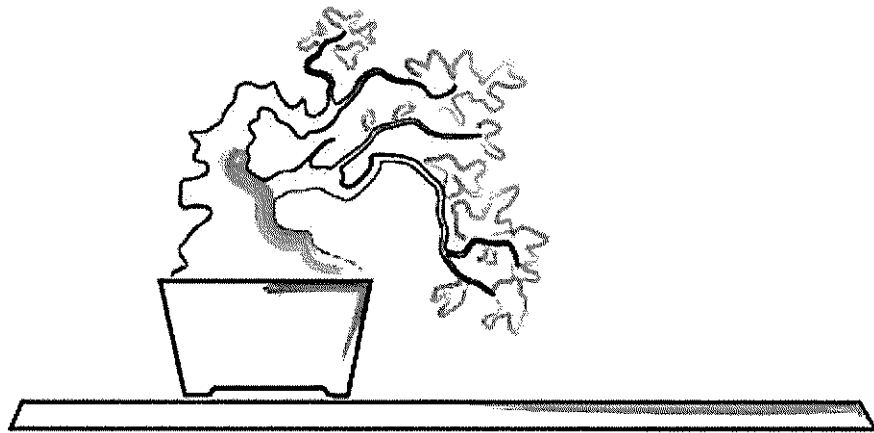
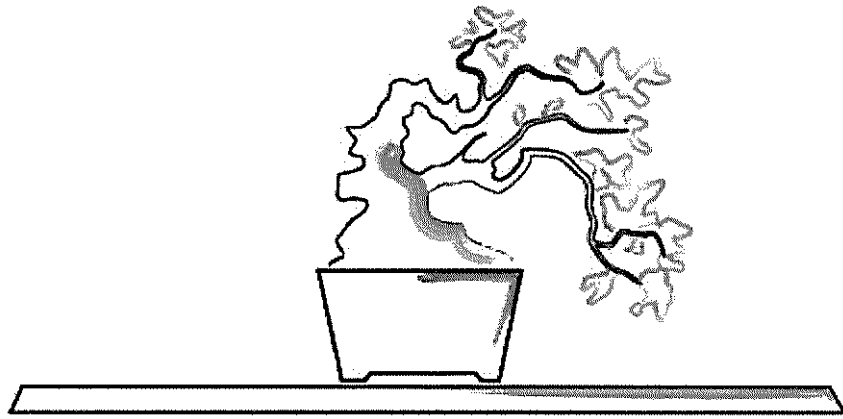
Paintings Survey

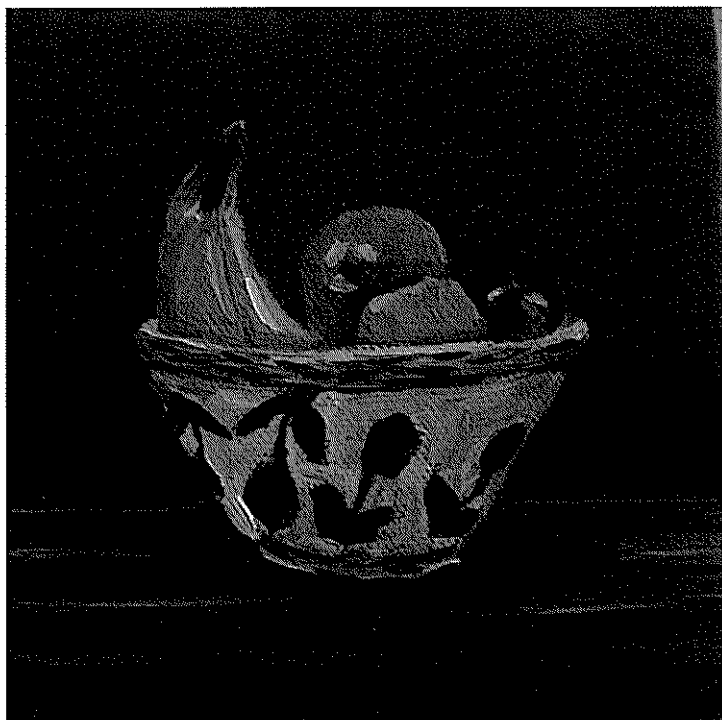
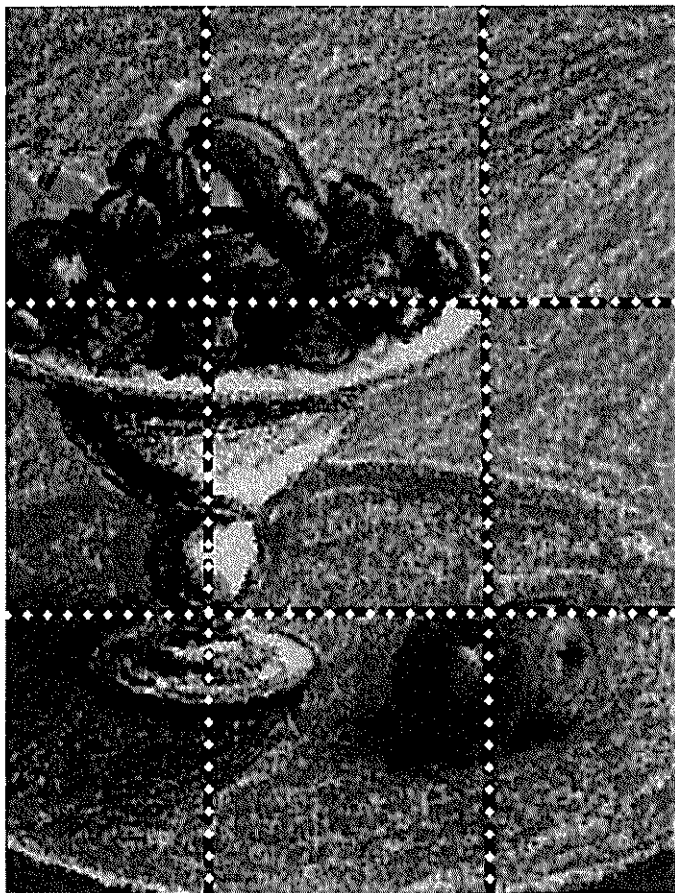


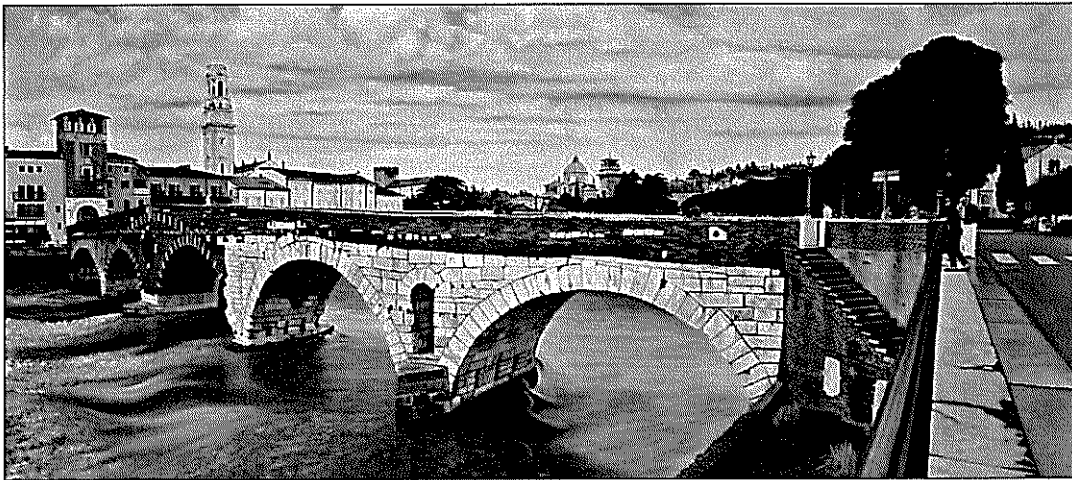
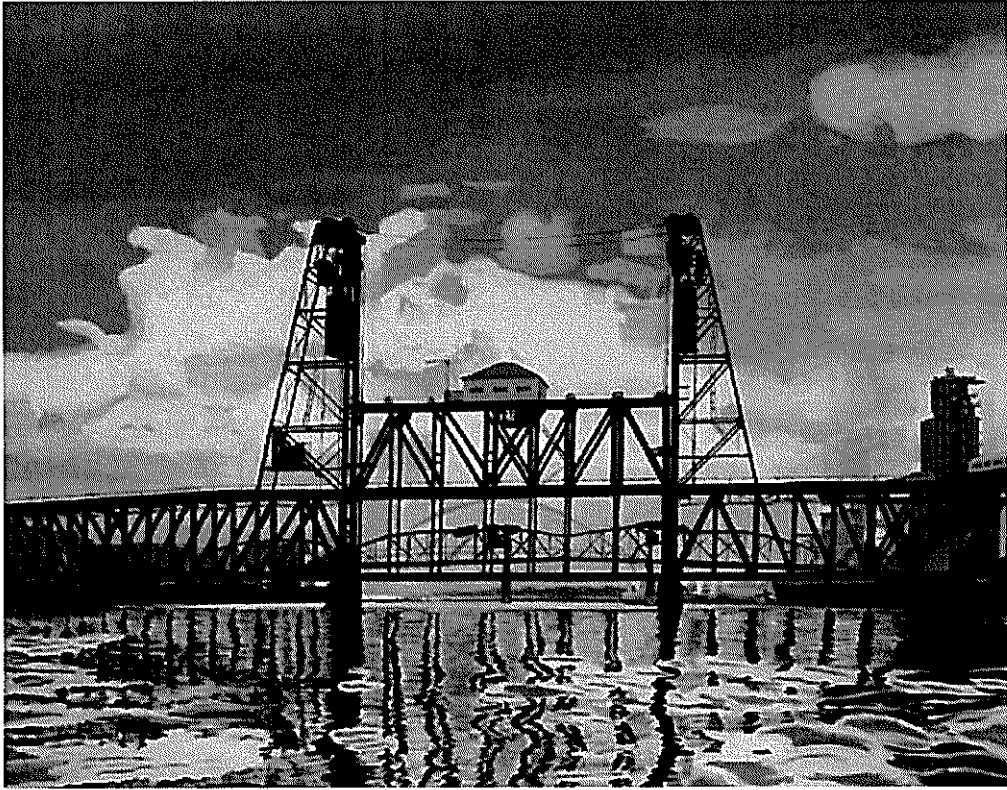


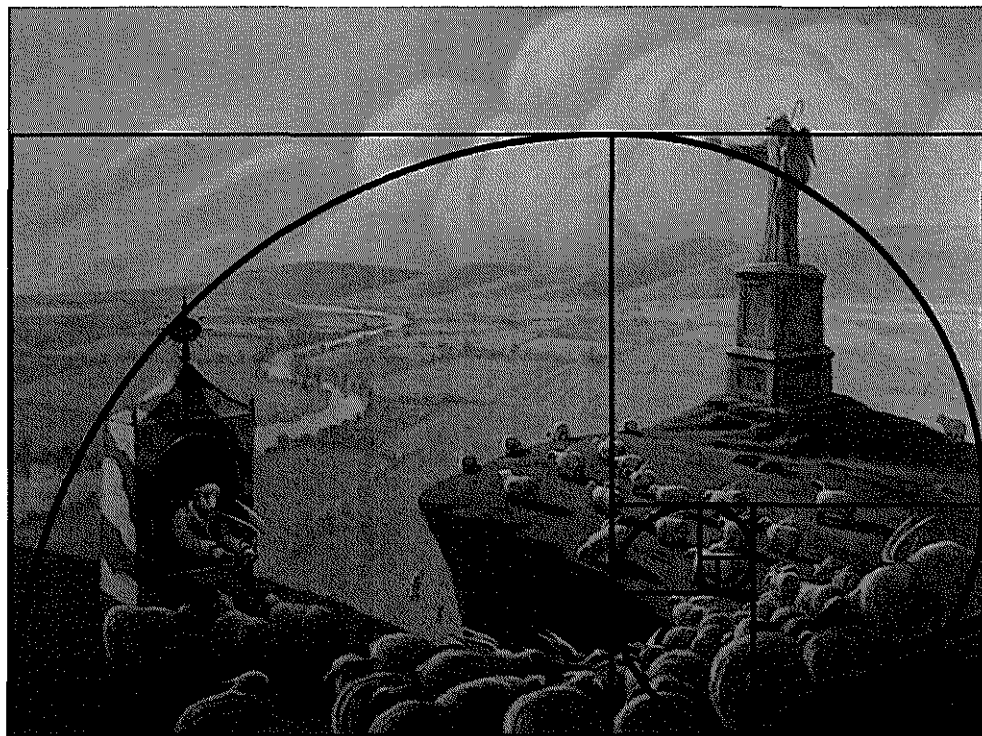
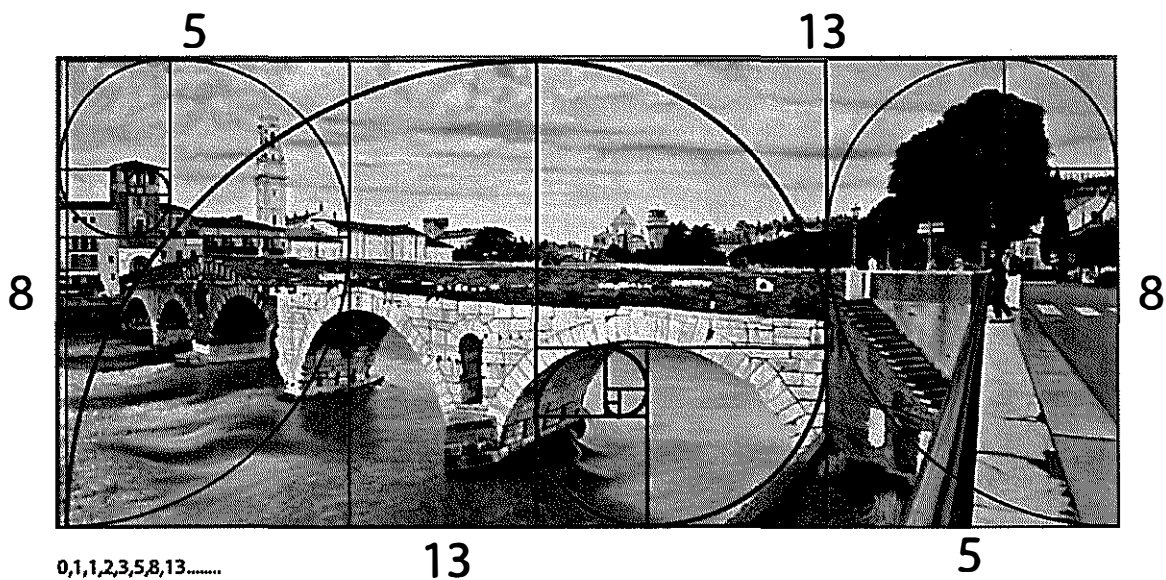


Bernie Rosage Jr. "Outstanding in its Field" 6x8" Acrylic on Panel











Line Test

activity sheet

Name _____

Investigate the Golden Ratio

Are we golden? Is the golden ratio somewhere in each of us? Form groups of four or five and use the table, directions, and skeletal diagrams to determine if you are golden.

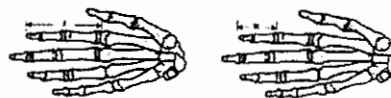
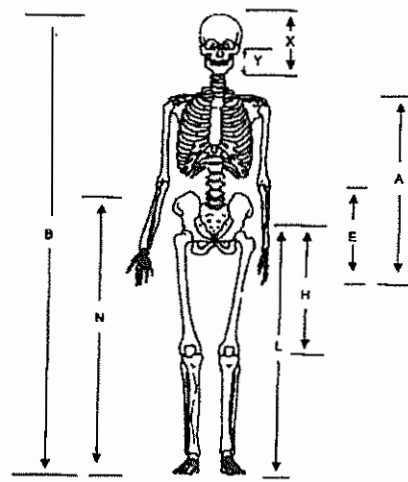
Step 1: Measure the height (B) and the navel height (N) of each member of your group. Calculate the ratio B/N. Record them in your table.

Step 2: Measure the length (F) of an index finger and the distance (K) from the fingertip to the big knuckle of each member of your group. Calculate the ratio F/K. Record them in your table.

Step 3: Measure the length (L) of a leg and the distance (H) from the hip to the kneecap of everyone in your group. Calculate and record the ratio L/H.

Step 4: Measure the length (A) of an arm and the distance (E) from the fingertips to the elbow of everyone in your group. Calculate and record the ratio A/E.

Step 5: Measure the length (X) of a profile (the top of the head to the level of the bottom of the chin) and the length (Y) (the bottom of the ear to the level of the bottom of the chin). Calculate and record the ratio X/Y.





Name	Express Each Ratio in Both its Fraction and Decimal Form				
	B/N	F/K	L/H	A/E	X/Y
1.					
2.					
3.					
4.					
5.					


Palindrome Routine


Chamber Camp 2014


Part I


1. 

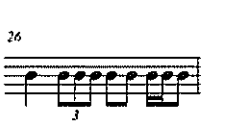
5 

6 

10 

17 

22 

26 

Palindrome Routine

Part II

2 

30 

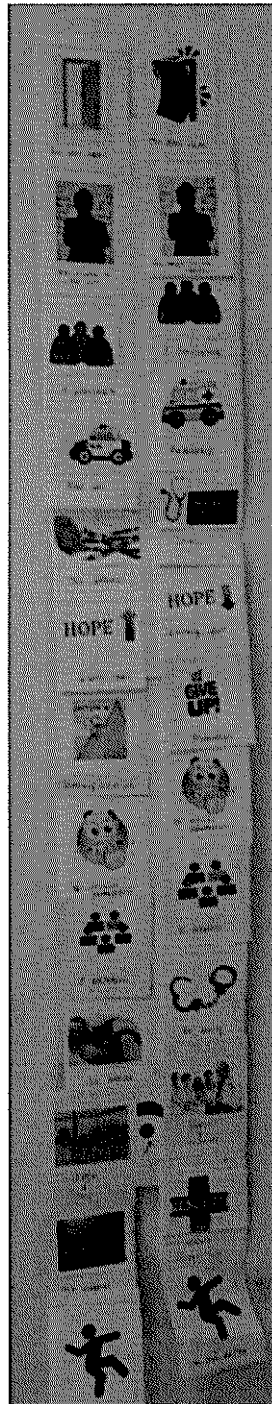
43 

51 

57 

61 

Palindrome Memory Game
Berg's Lulu Act II



(student example of a completed memory game)

Der Spiegel (The Mirror) Duet

ANON. *Allegro* ♩=120

attrib. to W.A. Mozart

The musical score for 'Der Spiegel (The Mirror) Duet' is presented in ten staves. It begins with a treble clef, a key signature of one sharp (F#), and a 3/4 time signature. The tempo is marked 'Allegro' with a metronome marking of ♩=120. The dynamic marking 'mf' (mezzo-forte) is indicated at the start. The score features a complex rhythmic pattern with many sixteenth and thirty-second notes, creating a shimmering, mirror-like effect. The piece concludes with a double bar line and a repeat sign.

Public Domain. Sequenced by Fred Nachbaur using NoteWorthy
Confused? Try playing this from opposite sides of a table.

Allegro

Canon No. 1. a 2 cancrizans The Musical Offering, J. S. Bach

The musical score for 'Canon No. 1. a 2 cancrizans' is shown in two staves. It is in G major and 3/4 time. The top staff features a simple, rhythmic melody, while the bottom staff provides a more complex, rhythmic accompaniment. The piece is a canon in two parts, where the second part is a rhythmic inversion of the first.

What does "Unleashing Music's Hidden Blueprint" Class 2014 sound like?

Caroline James Micheal Julia Emily

6 Madison Kira Teresa Abby Autumn Paige

12 Cara Stella Sam Laura Tommy Maya Alicia

19 Lexi Nathaniel Patrick Emma Miranda Christina Tiffany

26 Sarah Elena Natalie Dominic Shaun Ella

32 Ellie Amelia David

Final Survey

Fibonacci Series/Golden Mean

(Bartok, Debussy)

Poor

Best

Interesting	1	2	3	4	5
Possessed aesthetic beauty	1	2	3	4	5
Enhanced the Overall Piece	1	2	3	4	5
Distracted from the Piece	1	2	3	4	5
Enhance or Distract (circle one)					

Palindrome

(Machaut, Berg, Haydn, Hindemith)

Interesting	1	2	3	4	5
Possessed aesthetic beauty	1	2	3	4	5
Enhanced the Overall Piece	1	2	3	4	5
Distracted from the Piece	1	2	3	4	5
Enhance or Distract (circle one)					

Mirror

(Mozart)

Interesting					
Possessed aesthetic beauty	1	2	3	4	5
Enhanced the Overall Piece	1	2	3	4	5
Distracted from the Piece	1	2	3	4	5
Enhance or Distract (circle one)	1	2	3	4	5

Crab Canon/Möbius Strip

(Bach)

Interesting					
Possessed aesthetic beauty	1	2	3	4	5
Enhanced the Overall Piece	1	2	3	4	5
Distracted from the Piece	1	2	3	4	5
Enhance or Distract (circle one)	1	2	3	4	5

Fractal

(Bach)

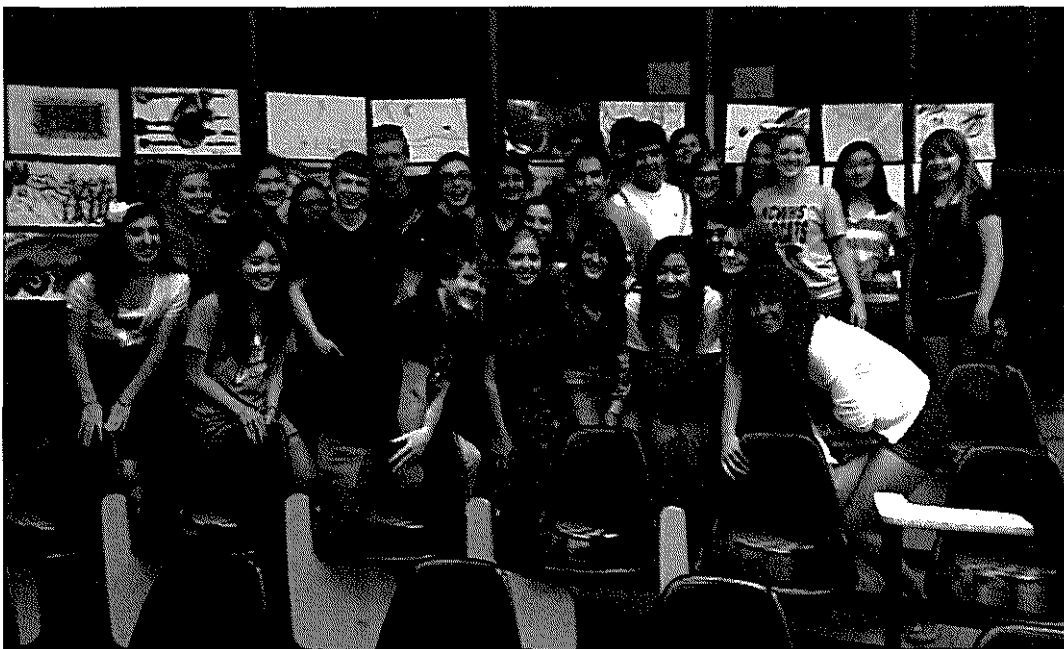
Interesting					
Possessed aesthetic beauty	1	2	3	4	5
Enhanced the Overall Piece	1	2	3	4	5
Distracted from the Piece	1	2	3	4	5
Enhance or Distract (circle one)	1	2	3	4	5

Which technique was most effective?

Photos



Palindrome Clapping Routine: At the *Children's Concert*, hosted by the IWU music camp, my class was able to perform a clapping sequence that we created. We actively lived a palindrome aurally through our clapping patterns and visually through out groupings of color. Here I demonstrate to the children in the audience how to clap a palindrome.



Last Day of Class: Final class photo in front of student's asymmetrical paintings from Day 1.

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