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LaSalle's Invariance Principle on Measure Chains

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Chapter 1

INTRODUCTION

In this paper we will be examining certain stability properties of autonomous systems. Suppose then that we are dealing with a system of the form

$$x' = f(x)$$

$$x(0) = x_0.$$

Here $f : B \longrightarrow R$, where $B \subseteq R^m$. Lyapunov (whose work is described in [7] and [12]) and LaSalle [8-12] have developed stability results for the continuous and discrete cases. The results given here include these as special cases. This greater generality is accomplished by means of the calculus on measure chains developed by Aulbach and Hilger [2, 6].

It was in 1892 that Lyapunov published his paper giving his "second method". The basic guiding principle was that we might be able to know something about the stability of the system from the form of the equations describing it. Specifically, the idea was that it would not be necessary to know the solutions of the equations involved. This is of course very useful since in most cases solutions are extremely difficult or even impossible to find. Lyapunov's insight was that if a function could be found with, among other properties, a negative rate of change along the solution of the system except in the equilibrium case, then disturbances from the equilibrium solution would return to that solution. (In the equilibrium case, the solution is constant.) The kind of function involved is called a Lyapunov function, and it is defined in such a way that it mimics the energy function. In fact, it was the energy function which originally inspired these ideas. There is an intuitive physical appeal about the assertion that systems that lose energy "fall" to an equilibrium state. And in many cases, the expression for energy ends up being our choice for Lyapunov function. The historical data above can be found in [5].

Lyapunov's method is extremely valuable, since it enables us to reach conclusions about stability without obtaining explicit solutions. The disadvantage is that finding an appropriate Lyapunov function can often be very difficult. In response to this fact, LaSalle produced an extension of Lyapunov's method in the early sixties. In this extension, LaSalle used the notion of limit sets (sets of limit points) and the notion of invariance (the property of certain sets whereby a given function takes elements in the set to elements in the set). By introducing these notions, LaSalle was able to show how Lyapunov functions could be defined less restrictively. His Invariance Principle is the invariance-and-limit sets version of Lyapunov's theorems describing his method. LaSalle has produced both discrete and continuous versions of his Principle.

The measure chain calculus was developed in response to the previously disunified state of analysis. Before the calculus on measure chains, results developed in the continuous calculus had to be independently confirmed in the discrete calculus, and vice versa, or else it was assumed without justification that results obtained in one case would apply in the other. Also, there was no method of dealing with functions defined on sets that were partially discrete, partially continuous. Thus it was that Drs. Aulbach and Hilger developed the concept of a measure chain, defined axiomatically, and derived a calculus for these chains. Specifically, they developed some preliminary items, such as an induction principle, and proceeded to invent notions of derivative, integral, and continuity. They proved, among other things, a measure chain version of the mean value theorem.

Chapter 2

STABILITY

2.1 Lyapunov's Second Method

The material in this section is based on [4]. Consider an arbitrary autonomous system, i.e. one of the form

$$y' = f(y)$$

where f and $\partial f / \partial y_j$, $j = 1, 2, ..., n$, are continuous in a region D of n-dimensional space. Assume that D contains the origin, and our goal shall be to find stability conditions for the zero solution. This is in fact no restriction at all, since a translation can always be effected if D does not contain the origin.

Consider a continuous scalar function $V(y)$ defined on some region Ω containing the origin. Recall that V is said to be positive definite on the set Ω if and only if $\forall y \in \Omega$ $V(y) > 0$ for $y \neq 0$ and $V(0) = 0$. Recall also that a scalar function $V(y)$ is said to be negative definite on the set Ω if and only if $-V(y)$ is positive definite on Ω. For example, in 3-space the function $V(y) = y_1^2 + y_2^2 + y_3^2$ is positive definite on the whole space. On the other hand, $V(y) = y_1^2$ is not positive definite, since it is zero everywhere on the $(y_2,\ y_3)$ plane.

We shall now define the derivative for the purposes of this discussion.

Definition 1 *The derivative of V with respect to y is $\dot{V}(y)$ =grad $V(y) \cdot f(y) = \frac{\partial V}{\partial y_1}(y) f_1(y) + \ldots + \frac{\partial V}{\partial y_n}(y) f_n(y)$, where $f_1, \ldots, f_n$ are the components of f.*

Example:

Consider the case (in the plane) of

$$y_1' = y_2$$

$$y_2' = -y_1 - 2y_2$$

and

$$V(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2).$$

In this case we obtain

$$\dot{V}(y_1, y_2) = \frac{1}{2} \cdot 2 \cdot y_1 \cdot y_2 + \frac{1}{2} \cdot 2 \cdot y_2(-y_1 - 2y_2)$$

$$= y_1 y_2 + y_2(-y_1 - 2y_2)$$

$$= -2{y_2}^2.$$

We must now define some notions of stability.

Definition 2 *We shall say that a zero solution to our system is stable if* $\forall \epsilon > 0\ \exists \delta(\epsilon, t_0) > 0$ *such that* $||x_0|| \leq \delta \Rightarrow ||\Phi(t; x_0, t_0)|| \leq \epsilon\ \forall t \geq t_0$. *Here* $\Phi(t; x_0, t_0)$ *represents the solution w.r.t. some initial values* x_0, t_0.

Definition 3 *Likewise, we shall say that the zero solution is asymptotically stable if it is stable and if* $\exists r(t_0) > 0$ *such that* $\forall \mu > 0\ \exists T(\mu, x_0, t_0)$ *such that* $||x_0|| \leq r(t_0) \Rightarrow ||\Phi(t; x_0, t_0)|| \leq \mu\ \forall t \geq t_0 + T$.

We are now ready to give Lyapunov's major results, which we shall present without proof. They are:

Theorem 1 *If there exists a scalar function $V(y)$ that is positive definite and for which $\dot{V}(y) \leq 0$ on some region Ω containing the origin, then the zero solution of $y' = f(y)$ is stable.*

Theorem 2 *If there exists a scalar function $V(y)$ that is positive definite and for which $\dot{V}(y)$ is negative definite on some region Ω containing the origin, then the zero soluion of $y' = f(y)$ is asymptotically stable.*

Example:

Consider the equation $u'' + g(u) = 0$, with g continuously differentiable for $|u| < k$, and $ug(u) > 0$ if $u \neq 0$. We can write this as a system of first-order equations:

$$y_1{}' = y_2$$

$$y_2{}' = -g(y_1)$$

Consider the function

$$V(y_1, y_2) = \frac{1}{2}{y_2}^2 + \int_0^{y_1} g(\sigma) d\sigma.$$

(This choice is motivated by physical considerations; it mimics the energy function. The first term represents kinetic energy;

the second represents potential energy.) The function is positive definite on

$$\{(y_1, y_2) : |y_1| < k, |y_2| < \infty\}.$$

Moreover

$$\dot{V}(y_1, y_2) = y_2 {y_2}' + g(y_1){y_1}' = y_2(-g(y_1)) + g(y_1)y_2) = 0.$$

Thus V satisfies the conditions of our Theorem 1, and we conclude the zero solution is stable.

Example:

Consider Lienard's equation

$$u'' + u' + g(u) = 0$$

which can be written as

$${y_1}' = y_2$$

$${y_2}' = -g(y_1) - y_2$$

where g is as in the previous example. Things proceed much as before if we take the same V, except that the derivative of V becomes $-{y_2}^2$. We can again conclude stability, but not asymptotic stability.

But in fact we rather imagine we have asymptotic stability. This is an example of where things can go wrong in Lyapunov's method.

2.2 LaSalle's Invariance Principle

2.2.1 Discrete Case

Introduction and Basic Notions

In this section, we turn to LaSalle's Invariance Principle. All material in this subsection and the next is based on [11] unless otherwise noted. LaSalle has developed this principle for both the continuous and discrete case. We will take the discrete first. In the discrete case, we have the simpler of the two situations. Solutions will always be bounded. As LaSalle says, "very little is required other than an understanding of convergence and continuity, and there are no troublesome questions concerning the existence and domain of definition of solutions."

Let the following conventions hold:

Let J be the set of all integers.

Let J_+ be the set of all nonnegative integers.

Let R^m be real m-space, with $||x||$ the Euclidean norm.

Let $x : J_+ \longrightarrow R^m$.

Let $x'(n) = x(n+1)$.

Let $\dot{x} = x' - x$.

Let $T : R^m \longrightarrow R^m$.

Consider then the initial value problem

$$x' = Tx, x(0) = x_0. \tag{2.1}$$

Its solution is of the form

$$x(n) = T^n(x_0).$$

where T^n is the nth iteration of T and $T^0 = I$, the identity mapping.

Definition 4 *We define a discrete dynamical system on R^m as a mapping $\pi : J_+ \times R^m \longrightarrow R^m$ such that $\forall n, k \in J_+$ and $\forall x \in R^m$,*

i) $\pi(0, x) = x$

ii) $\pi(n, \pi(k, x)) = \pi(n + k, x)$

iii) π *is continuous.*

As LaSalle puts it, "Every difference equation defines a dynamical system $\pi : \pi(n, x^0) = T^n x_0$, and, conversely, every discrete dynamical system has associated with it the difference equation (2.1), where $T(x) = \pi(1, x)$". A very good discussion of dynamical systems can be found in [13] and [14].

Some basic definitions:

Definition 5 *The distance of a point x from a set S is represented as $\rho(x, S)$ and is defined to be inf$\{||y - x|| : y \in S\}$.*

Definition 6 *The closure of a set S is represented as $\overline{S}$ and is defined to be $\{x : \rho(x, S) = 0\}$.*

Definition 7 *A set S is closed if $\overline{S} = S$ and open if its complement is closed.*

LaSalle's principle is based in large part on the notion of a limit set, the set of all subsequential limit points of $T^n x_0$. Under conditions of boundedness, this set will turn out to be invariant.

Definition 8 *We say that a point y is a limit point of $T^n x$ if there is a sequence of integers n_i such that $T^{n_i} x \longrightarrow y$ and $n_i \longrightarrow \infty$ as $i \longrightarrow \infty$. The limit set $\Omega(x)$ of the motion $T^n x$ from x is the set of all limit points of $T^n x$.*

Definition 9 *We say that a set H is positively invariant if $T(H) \subseteq H$, negatively invariant if $H \subseteq T(H)$, and invariant if $T(H) = H$.*

Definition 10 *We say that a closed invariant set H is invariantly connected if it is not the union of two nonempty disjoint closed invariant sets.*

Preliminary Results

We now turn to some preliminary results we shall need in the proof of LaSalle's Principle.

Theorem 3 *Every limit set $\Omega(x)$ is closed and positively invariant.*

PROOF:

By definition of a limit point, each point of distance 0 from $\Omega(x)$ is itself a member of $\Omega(x)$ (subsequences would approach such a point indefinitely). Hence $\Omega(x)$ is closed. Consider an arbitrary $y \in \Omega(x)$. By definition of $\Omega(x)$ there exists a sequence of integers n_i such that $n_i \to \infty$ and $T^{n_i}x \to y$ as $i \to \infty$. Since T is continuous, we have $T(T^{n_i}x) = T^{n_i+1}x \to Ty$. Thus $Ty \in \Omega(x)$. Thus $\Omega(x)$ is positively invariant. Thus every limit set $\Omega(x)$ is closed and positively invariant. ∎

Theorem 4 *If T^n is bounded for $n \in J_+$, then $\Omega(x)$ is nonempty, compact, invariant, is the smallest closed set that $T^n x$ approaches as $n \to \infty$, and is invariantly connected.*

PROOF:

Since $T^n x$ is bounded, $\Omega(x)$ cannot be empty. Moreover it must be bounded and, by the preceding result, closed. Thus by Heine-Borel it must be compact. Consider an arbitrary $y \in \Omega(x)$ and select n_i as in the proof of theorem 1. W.l.o.g. assume $T^{n_i-1}x$ converges, say to z. Then $T(T^{n_i-1}x) = T^{n_i}x \to Tz = y$. Thus $\Omega(x)$ is negatively invariant, and hence by Theorem 1 is invariant.

We shall show that $\Omega(x)$ is the smallest closed set that $T^n x$ approaches as $n \to \infty$. Since $\rho(T^n x, \Omega(x))$ is bounded, there is a sequence n_i such that $n_i \to \infty, T^{n_i}x$ converges, and $\rho(T^{n_i}x, \Omega(x))$ does not approach 0 as $i \to \infty$. This is a contradiction, so we conclude that $T^n x \to \Omega(x)$. Suppose now that $T^n x \to E$ as $n \to \infty$ and E is closed; then $\Omega(x) \subseteq E$. Thus $\Omega(x)$ is the smallest closed set that $T^n x$ approaches as $n \to \infty$.

We shall now show that $\Omega(x)$ is invariantly connected. Suppose that it were not. Then $\Omega(x)$ is the union of two disjoint closed nonempty invariant sets Ω_1 and Ω_2. These subsets will be compact: they are closed and, being subsets of a bounded set, bounded. There exist disjoint open sets U_1 and U_2 such that

$\Omega_1 \subset U_1$ and $\Omega_2 \subset U_2$. Now T is continuous and Ω_1 is compact, so that T is uniformly continuous on Ω_1. Thus there is an open set V_1 such that $\Omega_1 \subset V_1$ and $T(V_1) \subset U_1$. Since $\Omega(x)$ is the smallest closed set that $T^n x$ approaches, $T^n x$ must intersect both V_1 and U_2 an infinite number of times. But then there exists a covergent subsequence $T^{n_i}x$ that is not in either V_1 or U_2. Since $\Omega(x) \subseteq V_1 \cup U_2$, we have a contradiction, and hence $\Omega(x)$ is invariantly connected.

∎

The Principle Stated and Proved

We now turn to LaSalle's extension of Lyapunov's work. Let $V : R^m \longrightarrow R$. The derivative of V will be defined in the following way.

Definition 11 *The derivative has the form* $\dot{V}(x) = V(T(x)) - V(x)$.

(This is relative to our system.) The idea is that we could compute this derivative without a knowledge of solutions–that we could compute it purely from a knowledge of the right-hand side of our original equation, $x' = Tx$.

Definition 12 *We call* V *a Lyapunov function of* (2.1) *on* G *if* V *is continuous and* $\dot{V}(x) \leq 0 \ \forall x \in G$.

Two sets will need to be defined. The first is $E = \{x : \dot{V}(x) = 0, x \in \bar{G}\}$. The second is M, the largest invariant set in E.

Theorem 5 (LaSalle's Invariance Principle) *If (i) V is a Lyapunov function of (2.1) on G, and (ii) $x(n)$ is a solution of (2.1) bounded and in G for all $n \geq 0$, then there is a number c such that $x(n) \to M \cap V^{-1}(c)$ as $n \to \infty$.*

PROOF:

By our assumptions, $V(x(n))$ is nonincreasing with n and is bounded from below, so that there exists a real number c such that $V(x(n)) \to c$ as $n \to \infty$. Consider an arbitrary $y \in \Omega(x_0)$. There is a sequence n_i such that $n_i \to \infty$ and $x(n_i) \to y$. Since V is continuous, $V(x(n_i)) \to V(y) = c$. Thus, $\Omega(x_0) \subseteq V^{-1}(c)$. Since $\Omega(x_0)$ is invariant, $V(Ty) = c$ and $\dot{V}(y) = 0$. Therefore $\Omega(x_0) \subseteq E$. Therefore $\Omega(x_0) \subseteq M$.By the foregoing, it follows that $\Omega(x_0) \subseteq M \cap V^{-1}(c)$. Since $x(n) \to \Omega(x_0)$, $x(n) \to M \cap V^{-1}(c)$. ∎

Example:

Consider the system

$$x(n+1) = \frac{ay(n)}{1+x^2(n)}$$

$$y(n+1) = \frac{bx(n)}{1+y^2(n)}.$$

Let $V(x, y) = x^2 + y^2$. Then

$$\dot{V}(x, y) = (\frac{b^2}{(a+y^2)^2} - 1)x^2 + (\frac{a^2}{(1+x^2)^2} - 1)y^2.$$

In fact there are four cases to be considered here; we will deal with one partly, and one in detail. The first case is that of $a^2 < 1, b^2 < 1$. This reduces to Lyapunov's standard case. The second case is that of $a^2 \leq 1, b^2 \leq 1$ and $a^2 + b^2 < 2$. We may assume that $a^2 < 1$ and $b^2 = 1$. V is a Lyapunov function everywhere. Here $\dot{V} \leq (a^2 - 1)y^2$, and E is the x-axis. Also $T(x, 0) = (0, bx)$, so that M is the origin, and the origin is hence asymptotically stable. The remaining cases are $a^2 = b^2 = 1$, where we have approach to the origin or to a periodic motion, and $a^2 > 1, b^2 > 1$, where we do not have approach of any kind.

We now consider the question of stability; we require a differently formulated definition, which parallels that given previously.

Definition 13 *A set H is said to be stable if for each neighborhood U of H (an open set containing $\overline{H}$), there is a neighborhood W of H such that $T^n(W) \subseteq U$ for all $n \in J_+$.*

Definition 14 *A set H is an attractor if there is a neighborhood U of $\overline{H}$ such that $x \in U$ implies $T^n x \to \overline{H}$ as $n \to \infty$. H is said to be asymptotically stable if it is both stable and an attractor.*

Definition 15 *The region of attraction $R(H)$ of a set H is the set of all x such that $T^n x \to H$ as $n \to \infty$.*

We then have the following theorem, which we present without proof.

Theorem 6 *Let G be a bounded open positively invariant set. If V is a Lyapunov function of (2.1) on G, and $M \subseteq G$, then M is an attractor and $\overline{G} \subseteq R(M)$. If, in addition, V is constant on M, then M is asymptotically stable.*

2.2.2 Continuous Case

Introduction and Basic Notions

In the continuous case, things become somewhat more complicated. Most importantly, solutions can "blow up" in finite time. Also, solutions can go forwards or backwards in time. Because of these and certain other changes, it is necessary to introduce a notion of "precompactness", which requires that solutions be not only bounded but also that it have no limit points of a certain kind on the boundary of the domain of the right-hand side of the differential equation. In all other respects, however, development is parallel. Indeed, this very fact suggests that the Principle is ripe to be put on measure chains.

Let $f : G^* \longrightarrow R^n$, where G^* is an open set in R^n. Assume f to be continuous. Our differential equation will be of the form

$$\frac{dx}{dt} = \dot{x} = f(x), x(0) = x_0. \tag{2.2}$$

Solutions are exactly associated with dynamical systems (as explained in the last section, although the definition of dynamical system is different here as shall be seen). Thus, we may write the solution as $\pi(t, x_0)$. The solution to the above equation for the given initial value will be assumed unique.

It will be necessary to introduce two kinds of limit points here.

Definition 16 *Let* $\phi : (\alpha, \omega) \longrightarrow G^*$*, where* $-\infty \leq \alpha < 0 < \omega \leq \infty$*. A point p is said to be a positive (negative) limit point of* ϕ *if there is a sequence* $t_n \in (\alpha, \omega)$ *such that* $t_n \to \omega$ $(t_n \to \alpha)$ *and* $\phi(t_n) \to p$ *as* $n \to \infty$*. The set*

$\Omega(\phi)$ $(A(\phi))$ *of all positive (negative) limit points of* ϕ *is called the positive (negative) limit set of* ϕ.

Definition 17 *The interval* (α, ω) *is said to be maximal if* ω *finite implies* $\Omega(\phi) \cap G^*$ *is empty and if* α *finite implies* $A(\phi) \cap G^*$ *is empty.*

Our definition of a dynamical system will be as follows. We first introduce the idea of a local dynamical system.

Definition 18 *A local dynamical system is a mapping* π *with the following properties:*

i) Each solution $\pi(t,x)$ *of* (2.2) *satisfying* $\pi(0,x) = x$ *has for each* $x \in G^*$ *a maximal interval of definition* $I(x) = (\alpha(x), \omega(x))$, $-\infty \leq \alpha < 0 < \omega \leq \infty$.

ii) $\forall s \in I(x)$ $\forall t \in I(\pi(s,x))$, $t + s \in I(x)$ *and* $\pi(t, \pi(s,x)) = \pi(t+s, x)$.

iii) π *is continuous, i.e. if* $(t_n, x_n) \in I(x_n) \times G^*$ *and* $(t_n, x_n) \to (t, x) \in I(x) \times G^*$, *then* $\pi(t_n, x_n) \to \pi(t,x)$.

iv) $I(x)$ *is lower semicontinuous on* G^*, *i.e., if* $x_n \to x \in G^*$, *then* $I(x) \subseteq$ *lim inf* $I(x_n) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} I(x_n)$.

Definition 19 *A dynamical system is a local dynamical system such that* $\forall x \in G^*$ $I(x) = (-\infty, \infty)$.

It shall be necessary to introduce a notion of precompactness. This plays the same conceptual role as boundedness did in the discrete case.

Definition 20 *A solution $\pi(t,x)$ is said to be positively (negatively) precompact if it is bounded for all $t \in [0, \omega(x))$ $((\alpha(x), 0])$ and if it has no positive (negative) limit points on the boundary of G^*.*

Note that $A(x)$ and $\Omega(x)$ will represent the negative and positive limit sets of $\pi(t,x)$.

Definition 21 *A set $H \subseteq R^n$ is said to be positively (negatively) invariant if $x \in H \cap G^*$ implies $\pi(t,x) \in H$ for all $t \in [0, \omega(x))$ $(t \in [\alpha(x), 0])$. H is weakly invariant if it is positively and negatively invariant. H is invariant if $I(x) = (-\infty, \infty)$ for each $x \in H \cap G^*$.*

Preliminary results

We now prove two results necessary for the proof of the Principle. These are analogous to the two preliminary results in the discrete case.

Theorem 7 *Every positive limit set is closed and weakly invariant.*

PROOF:

Consider an arbitrary y such that $\rho(y, \Omega(x)) = 0$. By definition, $\inf\{||z - y|| : z \in \Omega(x)\} = 0$. Now $\forall z \in \Omega(x)\ \exists S(z) \subseteq (\alpha, \omega)$ such that $\inf\{||s - z|| : s \in S(z)\} = 0$. Thus $\inf\{||s - y|| : \exists z$ such that $s \in S(z)\} = 0$. Thus one can select from (α, ω) a sequence t_n such that $\pi(t_n, x_0) \to y$. Moreover one can do so such that $t_n \to \omega(x)$. Thus $\Omega(x) = \overline{\Omega(x)}$. (For $\Omega(x) \subseteq \overline{\Omega(x)}$ trivially, and the converse relation has been demonstrated by the above.) Thus $\Omega(x)$ is closed.

What remains is to show that $\Omega(x)$ is weakly invariant. Consider an arbitrary $y \in \Omega(x) \cap G^*$ and an arbitrary $t \in I(x)$. Now $I(x)$ is maximal, and *ex hypothesi* $\Omega(x) \cap G^*$ is nonempty, so that $\omega(x) = \infty$. Thus there is a sequence t_n such that $t_n \to \infty$ and $\pi(t_n, x) \to y$. By our condition of lower semicontinuity, we have that for all n sufficiently large, $t \in I(\pi(t_n, x))$. And $\pi(t, \pi(t_n, x)) = \pi(t + t_n, x) \to \pi(t, y)$ as $n \to \infty$. Thus $\pi(t, y) \in \Omega(x)$, and $\Omega(x)$ is positively invariant. Thus every positive limit set is closed and positively invariant. ∎

Theorem 8 *If $\pi(t, x)$ is positively precompact, then $\Omega(x)$ is in G^*, and is nonempty, compact, connected, invariant, and is the smallest closed set that*

$\pi(t, x)$ *approaches as* $t \to \infty$.

PROOF:

That $\Omega(x)$ is nonempty follows immediately from the premise. Since $\pi(t, x)$ can only have limit points in G^* or its boundary, it follows from precompactness that $\Omega(x)$ is in G^*. Since $\Omega(x)$ is closed by the previous result, and since the solution and hence $\Omega(x)$ is bounded, it follows by Heine-Borel that $\Omega(x)$ is compact. Now since $\Omega(x)$ is nonempty and in G^*, it follows that $\Omega(x) \cap G^*$ is nonempty, so that $I(x) = (-\infty, \infty)$. Since $\Omega(x)$ is weakly invariant by the previous result, it is invariant. Since $\Omega(x)$ consists exactly of those points which are the limits of the images under $\pi(t, x)$ of subsequences of R, it follows that $\pi(t, x)$ approaches $\Omega(x)$. If there were a closed subset E in $\Omega(x)$ which $\pi(t, x)$ approached, then $\Omega(x)$ would contain points of positive distance from E, which subsequences of $\pi(t, x)$ would nonetheless approach; this is a contradiction. Thus $\Omega(x)$ is the smallest closed set that $\pi(t, x)$ approaches as $t \to \infty$.

We shall now show that $\Omega(x)$ is invariantly connected. Suppose that it were not. Then $\Omega(x)$ is the union of two disjoint closed nonempty invariant sets Ω_1 and Ω_2. These subsets will be compact: they are closed and, being subsets of a bounded set, bounded. There exist disjoint open sets U_1 and U_2 such that $\Omega_1 \subset U_1$ and $\Omega_2 \subset U_2$. Now $\pi(t,x)$ is continuous and Ω_1 is compact, so that $\pi(t,x)$ is uniformly continuous on Ω_1. Thus there is an open set V_1 such that $\Omega_1 \subset V_1$ and $\pi(t,V_1) \subset U_1$. Since $\Omega(x)$ is the smallest closed set that $\pi(t,x)$ approaches, $\pi(t,x)$ must intersect both V_1 and U_2 an infinite number of times. But then there exists a covergent subsequence $\pi(t_i,x)$ that is not in either V_1 or U_2. Since $\Omega(x) \subseteq V_1 \cup U_2$, we have a contradiction, and hence $\Omega(x)$ is invariantly connected. ∎

Lyapunov functions

We now define Lyapunov functions for the continuous case. We take $V : G^* \longrightarrow R$.

Definition 22 *The derivative is of the form* $\dot{V}(x) = \frac{dV}{dt}$.

Definition 23 *Let* $V : G^* \longrightarrow R$, *and let* $G \subseteq G^*$. V *is a Lyapunov function of* (2.2) *on* G *if* V *is continuous and* $\forall x \in G$ $\dot{V}(x) \leq 0$.

The Principle

We begin by introducing some basic sets. These are all relative to a Lyapunov function V of (2.2) on G.

$E := \{x \in \overline{G} \cap G^* : \dot{V}(x) = 0\}$

M is the largest invariant set in E.

M^* is the largest weakly invariant set in E.

Theorem 9 (LaSalle's Invariance Principle (continuous case)) *Let V be a Lyapunov function of (2.2) on G, and let $x(t) = \pi(t, x_0))$ be a solution of (2.2) that remains in G for all $t \in [0, \omega(x_0))$. Then, for some $c, \Omega(x_0) \cap G^* \subseteq M^* \cap V^{-1}(c)$. If $x(t)$ is precompact, then $x(t) \to M \cap V^{-1}(c)$.*

PROOF:

Suppose $y \in \Omega(x) \cap G^*$. Then that set is nonempty, and hence $\omega(x) = \infty$. Then there is a sequence $t_i \in I(x)$ such that $t_i \to \infty$ and $x(t_i) \to \infty$ as $i \to \infty$. So by continuity of V we have $V(x(t_i)) \to V(y)$ as $t_i \to \infty$, $x(t_i) \to \infty$, and $i \to \infty$. But V is nonincreasing along $x(t)$, and thus $V(x(t)) \to V(y) =: c$.

We further conclude from this that $\forall y \in \Omega(x)\dot{V}(y) = 0$, and hence $\Omega(x) \subseteq E$. Moreover $\Omega(x)$ is weakly invariant, and hence invariant. Thus, $\Omega(x_0) \subseteq M \cap V^{-1}(c)$. Since $x(t) \to \Omega(x_0)$ as $t \to \infty$, $x(t) \to M \cap V^{-1}(c)$ as $t \to \infty$. ∎

We have as a corollary

Corollary 10 *Let V be a Lyapunov function of* (2.2) *on G and let $x(t)$ be a precompact solution of* (2.2) *that remains in G for all $t \geq 0$. If the points of intersection of M (or E) with $V^{-1}(c)$ are isolated for each c, then $x(t)$ approaches an equilibrium point of* (2.2) *as $t \to \infty$.*

We now consider the notion of stability.

Definition 24 *A compact set $H \subseteq G^*$ is said to be stable, if given a neighborhood U of H, there is a neighborhood W of H such that $x \in W$ implies $\pi(t, x) \in U$ for all $t \geq 0$.*

Definition 25 *A compact set $H \subseteq G^*$ is an attractor if there is a neighborhod U of H such that $x \in U$ implies $\pi(t, x) \to H$ as $t \to \infty$. If H is both stable and an attractor, H is said to be asymptotically stable.*

Definition 26 *The region of attraction $R(H)$ of a set H in G^* is the set of all $x \in G^*$ such that $\pi(t, x) \to H$ as $t \to \infty$.*

Thus we have the following result, which we present without proof.

Theorem 11 *Let G be a positively invariant open set in G^* with the property that each solution starting in G is bounded and has no positive limit points on the boundary of G. If V is a Lyapunov function of (2.2) on G, $M_0 := \overline{M \cap G} \subseteq G$, and M_0 is compact, then M_0 is an attractor and $G \subseteq R(M_0)$. If in addition, V is constant on the boundary of M_0, then M_0 is asympototically stable.*

Chapter 3

THE CALCULUS ON MEASURE CHAINS

3.1 The Axioms

The material in this chapter is based on [6], except where otherwise noted. The axiomatic development of measure chains runs as follows.

Axiom 1 *There exists an ordering relation* $\leq$ *on the time scale* T *which satisfies the following conditions:*

i) reflexivity ($\forall t \in T, t \leq t$)

ii) transitivity ($\forall r, s, t \in T, r \leq s$ *and* $s \leq t \Longrightarrow r \leq t$)

iii) antisymmetricity ($\forall r, s \in T, r \leq s$ *and* $s \leq r \Longrightarrow r = s$.)

iv) totality ($\forall r, s \in T, r \leq s$ *or* $s \leq r$).

In general, this will be the standard "less than or equal to" relation, regardless of the time scale under consideration.

Axiom 2 *T is conditionally complete: each subset of T bounded above has a least upper bound.*

Axiom 3 *There exists a function $\mu : T \times T \longrightarrow R$ such that $\forall r, s, t \in T$ we have*

i) $\mu(r, s) + \mu(s, t) = \mu(r, t)$

ii) $r > s \Longrightarrow \mu(r, s) > 0$

iii) μ *is continuous.*

The natural example here is the directed distance function $\mu(r, s) = r - s$. This gives the standard discrete calculus on $hZ := \{hz : z \in Z\}$ for any real number h. The measure gives the standard continuous calculus on R.

3.2 Jump operators

A useful concept will be that of the jump operator. Thus:

Definition 27 *The forward jump operator on T is the function $\sigma : T \longrightarrow T$ such that*

$$\sigma(t) = inf\{s \in T : s > t\}.$$

Definition 28 *The backward jump operator on T is the function $\rho : T \longrightarrow T$ such that*

$$\rho(t) = sup\{s \in T : s < t\}.$$

Intuitively, the one takes us to the "next" element in the set (if such exists) and the other takes us to the "previous" element in the set (if such exists). If no "next" element exists, then $\sigma(t) = t$; similarly for ρ. Thus for hZ, $\sigma(hz) = h(z+1)$ and $\rho(hz) = h(z-1)$. For $R, \sigma(t) = \rho(t) = t$.

Definition 29 *We say that an element is right-dense if $\sigma(t) = t$; we say it is right-scattered if $\sigma(t) > t$. We say that an element is left-dense if $\rho(t) = t$; we say that an element is left-scattered if $\rho(t) < t$.*

Thus each element of hZ is right- and left-scattered; each element of R is right- and left-dense.

3.3 Some Known Results

(Note: All results in this section will be presented without proof.) From the first two axioms we can derive the Heine-Borel theorem on measure chains:

Theorem 12 *A set in a measure chain T is compact if and only if it is closed and bounded.*

Here it should be understood that all topological statements are made w.r.t. the standard order topology. This is the topology usually assumed for R; in hZ this is the discrete topology.

We can also demonstrate an intermediate value theorem:

Theorem 13 (Intermediate Value Theorem) *Given the continuous mapping $f : [r, s] \longrightarrow R$, with $r, s \in T$, which fulfills the condition $f(r) < 0 < f(s), \exists \tau \in [r, s]$ such that*

$$(f(\tau))(f(\sigma(\tau))) \leq 0.$$

There is also an induction principle:

Theorem 14 (Principle of Induction) *Assume that for a family of statements $A(t)$, $t \in [\tau, \infty) \subseteq T$, the following conditions are fulfilled:*

i) $A(\tau)$

ii) for each right-scattered $t \in T$ we have $A(t) \Longrightarrow A(\sigma(t))$

iii) for each right dense $t \in T$ there is a neighborhood U such that $A(t)$ $\Longrightarrow A(s)$ for each $s \in U$ with $s > t$

iv) for each left dense $t \in T$ we have ($A(s)$ $\forall s$ such that $s < t$) $\Longrightarrow A(t)$

Then $A(t)$ is true $\forall t \in [\tau, \infty)$.

Note that for $T = N$, (3) and (4) are trivially satisfied, and (2) becomes "For each $t \in T$ we have $A(t) \Longrightarrow A(\sigma(t))$". Thus the above principle becomes the standard (weak) induction principle on natural numbers.

3.4 Differentiation

We now introduce the concept of a derivative.

Definition 30 *Consider a function $f : T \longrightarrow X$, where X is some Banach space. At a point $t \in T$ we say that f has the derivative $f^{\Delta}(t) \in X$ if $\forall \epsilon > 0$ $\exists$ a neighborhood U of t such that $\forall s \in U$*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t) \cdot \mu(\sigma(t), s)| \leq \epsilon |\mu(\sigma(t), s)|.$$

f is called differentiable in t if f has exactly one derivative in t.

We define $T^\kappa := \{t \in T : t \text{ is nonmaximal or } t \text{ is left-dense}\}$. Thus T^κ is identical with T unless T has an isolated upper endpoint t^*, in which case $T^\kappa = T - \{t^*\}$.

Definition 31 *We say that f is pre-differentiable on T with region of differentiation D if $D \subseteq T^\kappa, T^\kappa - D$ is countable and contains no right-scattered elements of T, f is continuous on T and differentiable in each t in D.*

This may seem strange and arbitrary; the definition was created in order to prove a theorem which gives existence of anti-derivatives.

Example:

From [3]: consider the case of $T := \{a^m : m \in Z\} \cup \{0\}$. Here $\sigma(t) = \inf\{a^n : n \in [m+1, \infty)\} = a^{m+1} = a(a^m) = at$ where $t \neq 0$. Moreover, $\sigma(0) = 0$. Thus $\forall t \in T$ we have $\sigma(t) = at$ and $\rho(t) = t/a$. Define $\mu(s,t)$ to be $s - t$. Thus 0 is a right-dense minimum and every other point in T is both left- and right-scattered. For a function $f : T \longrightarrow R$ we must find for each $\epsilon > 0$ a neighborhood U of t such that $\forall s \in U$

$$f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|.$$

For $t \neq 0$, let U $= \{t\}$. Then we have

$$|f(\sigma(t)) - f(t) - f^{\Delta}(t)(\sigma(t) - t)| \leq \epsilon|\sigma(t) - t|.$$

Now since this must hold for all ϵ, we have

$$|f(\sigma(t)) - f(t) - f^{\Delta}(t)(\sigma(t) - t)| \leq 0,$$

and hence

$$|f(\sigma(t)) - f(t) - f^{\Delta}(t)(\sigma(t) - t)| = 0.$$

Thus

$$f(\sigma(t)) - f(t) - f^{\Delta}(t)(\sigma(t) - t) = 0,$$

and hence

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{f(at) - f(t)}{(a-1)t}, \end{aligned}$$

where $t = a^m$. Moreover,

$$f^{\Delta}(0) = \frac{|f(0) - f(s)|}{|s|}.$$

Example:

From [3]: let H_n be the harmonic numbers, so that $H_0 = 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ $\forall n \in N$. Then consider the time scale $T := \{H_n : n \in N_0\}$. Define $\mu(s,t)$ to be $s - t$. Then all points are both left- and right-scattered. We have $\sigma(H_n) = H_{n+1} \forall n \in N_0$, and $\rho(H_n) = H_{n-1}$ $\forall n \in N$, and $\rho(H_0) = 0$. For a function $f : T \longrightarrow R$ we must find for each $\epsilon > 0$ a neighborhood U of H_n such that $\forall s \in U$

$$|f(\sigma(H_n)) - f(s) - f^{\Delta}(H_n)(\sigma(H_n) - s)| \le \epsilon|\sigma(H_n) - s|.$$

Let U $= \{H_n\}$. Then we have

$$|f(\sigma(H_n)) - f(H_n) - f^{\Delta}(H_n)(\sigma(H_n) - H_n)| \le \epsilon|\sigma(H_n) - H_n|.$$

Now since this must hold for all ϵ, we have

$$|f(\sigma(H_n)) - f(H_n) - f^{\Delta}(H_n)(\sigma(H_n) - H_n)| \le 0,$$

and hence

$$|f(\sigma(H_n)) - f(H_n) - f^{\Delta}(H_n)(\sigma(H_n) - H_n)| = 0.$$

Thus

$$f^{\Delta}(H_n) = \frac{f(H_{n+1}) - f(H_n)}{H_{n+1} - H_n}$$

$$= (n+1)(f(H_(n+1) - f(H_n)).$$

Consider then the initial value problem

$$x^{\Delta} = \alpha x, x(0) = 1,$$

where $\alpha \in R$. The solution is given by

$$x(H_n) = \binom{n+\alpha}{n}$$

as can be seen by

$$x^{\Delta}(H_n) = (n+1)(f(H_{n+1} - f(H_n))$$

$$= (n+1)(\binom{n+1+\alpha}{n+1} - \binom{n+\alpha}{n})$$

$$= (n+1)\binom{n+\alpha}{n+1}$$

$$= (n+1)\frac{(n+\alpha)...\alpha}{(n+1)!}$$

$$= \frac{(n+\alpha)...\alpha}{n!}$$

$$= \alpha\binom{n+\alpha}{n}$$

$$= \alpha x(H_n).$$

Something needs to be said here about the chain rule. This case poses certain difficulties (see [1]). For suppose we have a function $f : T \longrightarrow T'$ and a function $g : T' \longrightarrow R$. What we should like to say, patterning our formula after the chain rule for the continuous case, is

$$(f \circ g)^{\Delta} = (f^{\Delta} \circ g)(g^{\Delta}(t)).$$

But in fact a problem arises here. For

$$|f \circ g(\sigma(t)) - f \circ g(s) - (f \circ g)^{\Delta}(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for s in some appropriately chosen U_1. But

$$|f(\sigma(g(t))) - f(g(s)) - f^{\Delta}(\sigma(g(t)) - g(s))| \leq \epsilon|\sigma(g(t)) - g(s)|$$

and

$$|g(\sigma(t)) - g(s) - g^{\Delta}(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|.$$

But it may be the case that $g(\sigma(t)) \neq \sigma(g(t))$. Thus the chain rule cannot be justified in this form. It is for this reason that we introduce the generalized jump operator (explained in [1]). A generalized jump operator is a function α mapping T into itself. The α-derivative is defined as the derivative is, substituting 'α' for 'δ'.

Theorem 15 *Let T and X be time scales with generalized jump operators α and β respectively. Let $g : T \longrightarrow X$ and $w : x \longrightarrow R$. Suppose that t is a* *point which is not an isolated extremum and is such that* $g(\alpha(t)) = \beta(g(t))$. *If* $g^\alpha(t)$ *and* $w^\beta(g(t))$ *exist, then*

$$(w \circ g)^\alpha = (w^\beta \circ g) g^\alpha$$

at t.

This theorem and its proof can be found in [1].

There is also a mean value theorem for measure chains.

Theorem 16 (Mean Value Theorem) *Let the mappings* $f : T \longrightarrow X, g : T \longrightarrow R$, *be predifferentiable with* D, *and assume that* $|f^\Delta(t)| \leq g^\Delta(t), t \in D$. *Then for* $r, s \in T, r \leq s$,

$$|f(s) - f(r)| \leq g(s) - g(r).$$

3.5 Rd-continuity and Integration

We now introduce the notion of rd-continuity.

Definition 32 *A function* g *is called rd-continuous if it is continuous in each right-dense or maximal* t *in* T *and the left sided limit exists in each left-dense* t.

Another important notion is that of a regulated function.

Definition 33 *A function g is called regulated if in each left-dense t in T the left sided, and in each right-dense t in T the right sided limit exists.*

These notions will be useful in the development of the integral.

The following result is due to Hilger; we present it without proof.

Theorem 17 *Let $\tau \in T, x \in X$, and a regulated mapping $g : T^{\mathrm{K}} \longrightarrow X$ be given. Then there exists exactly one function f, the pre-antiderivative, which is predifferentiable and fulfills the identities $f^{\Delta}(t) = g(t)$ for $t \in D$ and $f(\tau) = x$.*

The development of the integral has not proceeded along the lines of measure theory (i.e. the Riemann integral). The definition is rather as follows:

Definition 34 *For a regulated funtion $g : T^{\mathrm{K}} \longrightarrow X$ let $f : T \longrightarrow X$ be the pre-anti-derivative. Then*

$$\int_s^r g(t)\,\Delta t := f(s) - f(r) \in X.$$

Definition 35 *Let $g : T^{\mathrm{K}} \longrightarrow X$. The mapping $f : T \longrightarrow X$ is called antiderivative of g on T if it is differentiable on T and satisfies $f^{\Delta}(t) = g(t)\ \forall t \in T^{\mathrm{K}}$.*

We have the following result, from Hilger; we present it without proof.

Theorem 18 *If $g : T^{\mathrm{K}} \longrightarrow X$ is rd-continuous, then g has the antiderivative f, where $f(t) = \int_{\tau}^{t} g(s)\,\Delta s$.*

Chapter 4

LASALLE'S INVARIANCE PRINCIPLE ON MEASURE CHAINS

4.1 Introduction and Basic Notions

Our method of proceeding will essentially be that outlined by LaSalle. Most definitions will appear as natural extensions of his own definitions. The main difficulty will arise in the notion of a derivative, specifically that of the Lyapunov function V. The definition of a Lyapunov function puts certain conditions on the derivative along a solution. This requires the chain rule, with the attendant complications introduced earlier.

We begin by considering a time scale T which contains 0. We consider a function $x : T \longrightarrow R$ and a function $f : G^* \longrightarrow R$, where G^* is the largest open set in T', the time scale which is the image of T under x. The sort of "delta equation" we shall look at, then, will have the form

$$x^{\Delta} = f(x), x(0) = x_0. \tag{4.1}$$

We assume f to be continuous, and that the solution $\pi(x_0, t)$ is unique.

Definition 36 *Consider a function $\phi : (\alpha, \omega) \cap T \longrightarrow G^*$, where $\alpha \in T, \omega \in T$, and $-\infty \leq \alpha < 0 < \omega \leq \infty$. A point $p \in T$ is a positive (negative) limit point of ϕ if $\omega \in T$ ($\alpha \in T$) and there is a sequence $t_n \in (\alpha, \omega) \cap T$ such that $t_n \rightarrow \omega$ ($t_n \rightarrow \alpha$) and $\phi(t_n) \rightarrow p$ as $n \rightarrow \infty$. The set $\Omega(\phi)$ ($A(\phi)$) of all positive (negative) limit points of ϕ is called the positive (negative) limit set of ϕ.*

Definition 37 *The interval $(\alpha, \omega) \cap T$ is maximal if $\omega < \infty$ (or less than the maximal point of T, if such exists) implies $\Omega(\phi) \cap G^* = \{\}$ and $\alpha > -\infty$ (or greater than the minimal point of T, if such exists) implies $A(\phi) \cap G^* = \{\}$.*

Definition 38 *A local dynamical system on a time scale* T *is a mapping* $\pi : T \times R^m \longrightarrow R^m$ *such that for some maximal interval of definition* $I(x) = (\alpha(x), \omega(x)) \cap T$

i) $\pi(0, x) = 0$ *holds for some maximal interval of definition* $I(x) = (\alpha(x), \omega(x))$, $-\infty \leq \alpha < 0 < \omega \leq \infty$ *for each* $x \in G^*$.

ii) $\forall s \in I(x)\ \forall t \in I(\pi(s, x))$, *we have that* $s + t \in T$ *implies* $s + t \in I(x)$ *and that* $\pi(t, \pi(s, x)) = \pi(s + t, x)$.

iii) π *is continuous, i.e. if* $(t_n, x_n) \in G^* \times I(x_n)$ *and* $(t_n, x_n) \to (t, x) \in G^* \times I(x)$, *then* $\pi(t_n, x_n) \to \pi(t, x)$.

iv) $I(x)$ *is lower semicontinuous on* G^*, *i.e., if* $x_n \to x \in G^*$, *then* $I(x) \subseteq$ *lim inf* $I(x_n) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} I(x_n)$.

Definition 39 *A dynamical system on a time scale* T *is a local dynamical system on* T *with* $I(x) = T\ \forall x \in G^*$.

Solutions to (4.1) correspond to particular dynamical systems on T and vice versa.

Definition 40 *A solution* $\pi(t, x)$ *is positively (negatively) precompact if it is bounded for all* $t \in [0, \omega(x)) \cap T$ $(t \in (\alpha(x), 0] \cap T)$ *and has no positive (negative) limit points on the boundary of* G^*.

Since $\omega(x)$ is maximal, if $\pi(t,x)$ is positively precompact then $\omega(x) = \infty$. We use $\Omega(x)$ and $A(x)$ to denote the positive and negative limit sets respectively of $\pi(x,t)$.

Definition 41 *For* (4.1), *a set $J \subseteq R^n$ is said to be positively (negatively) invariant if $x \in H \cap G^*$ implies $\pi(t,x) \in J$ for all $t \in [0,\omega(x)) \cap T$ $((\alpha(x),0] \cap T)$. H is said to be weakly invariant if it is positively and negatively invariant. If, in addition, $I(x) = T$ for all $x \in H \cap G^*$, H is said to be invariant.*

If H is precompact relative to G^* and weakly invariant, then it is invariant.

4.2 Preliminary Theorems

We now introduce the usual theorems, with proof.

Theorem 19 *Every positive limit set is closed and weakly invariant.*

PROOF:

Closedness follows as before. For weak invariance, consider the fact that the set of limit sets we encounter for measure chains will be the same as that encountered in the continuous case. If solutions starting in limit sets stay there in the continuous case, they plainly cannot do otherwise for measure chains, which are after all subsets of R. ∎

Theorem 20 *If $\pi(t,x)$ is positively precompact, then $\Omega(x)$ is in G^*, and is nonempty, compact, connected, invariant, and is the smallest closed set that $\pi(t,x)$ approaches as $t \to \infty$.*

The proof is the same as in the continuous case.

4.3 Lyapunov functions

Let $V : T' \longrightarrow R$. Take α to be the generalized jump operator on T and β to be that of T'.

Definition 42 *The derivative here has the form $\dot{V}(x) = V^{\beta}(x)$*

Letting $x(t) = \pi(t,x)$, we see that the derivative of V w.r.t. $t \in T$ along the solution becomes

$$\dot{V}(x(t)) = V^{\alpha}(x(t)) = V^{\beta}(x) \cdot x^{\alpha}(t).$$

Definition 43 *Let $V : T' \longrightarrow R$, and let G be any subset of G^*. V is said to be a Lyapunov function of (4.1) on G if V is continuous, and $\dot{V}(x) \leq 0 \; \forall x \in G$.*

4.4 LaSalle's Invariance Principle on Measure Chains

We proceed essentially as in the continuous case. Relative to a Lyapunov function V of (4.1) on some $G \subseteq G^*$ we say that $E = \{x \in \overline{G} \cap G^* : \dot{V}(x) = 0\}$. We say that M is the largest invariant set in E and that M^* is the largest weakly invariant set in E.

Theorem 21 (LaSalle's Invariance Principle on Measure Chains) *Let V be a Lyapunov function of* (4.1) *on some $G \subseteq G^*$, and let $x(t) = \pi(t, x_0)$ be a solution of* (4.1) *that remains in G for all $t \in [0, \omega(x_0)) \cap T$. Then, for some c, $\Omega(x_0) \cap G^* \subseteq M^* \cap V^{-1}(c)$. If $x(t)$ is precompact, then $x(t) \to M \cap V^{-1}(c)$.*

PROOF:

Assume that $y \in \Omega(x_0) \cap G^*$. Then $\omega(x_0)$ is ∞ or the maximal point of T, if such exists. Let t^{**} stand for this maximal point or for infinity, which ever is appropriate. Thus there is a sequence t_n such that $x(t_n) \to y$ and $t_n \to t^{**}$ as $n \to \infty$. By continuity of V we have $V(x(t_n)) \to V(y)$ as $n \to \infty$. In fact $V(x(t))$ is nonincreasing w.r.t. t, so that we have $V(x(t)) \to V(y) =: c$.

Since $V(x(t))$ can converge to at most one point, we have $V(y) = c$ for all $y \in \Omega(x_0) \cap G^*$. Now $y \in G^* \cap \overline{G}$, and

$$|V(\beta(y)) - V(s)| \leq \epsilon|\beta(y) - s|$$

for s in some neighborhood $U(\epsilon)$. For regardless of what convergent sequence we select from U we arrive at another limit point, whose value under V is that of $V(y)$, namely c. Thus $\Omega(x_0) \cap G^*$ is in E. Moreover it is weakly invariant. Thus $\Omega(x_0) \cap G^* \subseteq M^* \cap V^{-1}(c)$. If $x(t)$ is precompact, $\Omega(x_0) \cap G^*$ is invariant. Thus $\Omega(x_0) \subseteq M$; thus $\Omega(x_0) \cap G^* \subseteq M$; thus $\Omega(x_0) \cap G^* \subseteq M \cap V^{-1}(c)$. Hence $x(t) \to M \cap V^{-1}(c)$. ∎

Chapter 5

CONCLUSION

Thus we can see how LaSalle's results may be generalized in such a way that the continuous and discrete cases are considered together, along with other important cases. This is done through the calculus on measure chains. LaSalle's Invariance Principle, an extension of Lyapunov's method, has been justified in this wider context.

The next step would be to develop results on stability as LaSalle has done. Ideally, work should be done concerning vector Lyapunov functions. These are discussed in [11]. Also, investigations should be carried out into the case of nonautonomous systems. These are systems of the form

$$\dot{x} = f(x, t).$$

A discussion of this more difficult case can be found in Zvi Artstein's appendix to [11].

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