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Chris Pelto '97
Illinois Wesleyan University

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Chris Pelto, Keith Coates, Narendra Jaggi
Illinois Wesleyan University

Abstract

The relatively mundane damped harmonic oscillator is found to exhibit interesting motion once under the influence of both a velocity dependent and a Coulombic frictional damping force. Data for the decay of the amplitude as a function of time were collected on a specially prepared torsional oscillator with a variable electromagnetic damping mechanism. An analytical solution of the appropriate equation of motion was obtained by the method of Laplace transforms. In both the limits of zero Coulombic friction and zero velocity damping, the solution reduces to the well-known answers to the problem. the solution, when plotted with the correct parameters, fits the numerical solution very well. The solution also shows excellent quantitative agreement with the experimental data.

The Torsional Oscillator

A torsional harmonic oscillator was used to study harmonic motion under the desired conditions. The system consists of a copper wheel which is free to rotate about an axle through its center. The restoring force is supplied by a linear coiled spring connected to the wheel about this axis of rotation. This force supplied by the spring is described by Hooke's Law, which says the magnitude of the force is linearly proportional to the displacement of the wheel from equilibrium.

The velocity dependent force is supplied via an electromagnet positioned such that its poles provide a magnetic field at the bottom of the wheel. When the wheel is allowed to rotate, the changing area of the copper in this field causes eddy currents to arise in the metallic wheel. The presence of moving currents then causes a force to arise which opposes the motion of the wheel. The retarding force is linearly dependent on the velocity with which the wheel is moving. The magnitude of the force can be controlled by varying the amount of current passing through the coils of the electromagnet.

This velocity dependent damping force varies linearly with the amount of current flowing in the coils[1].

A constant damping force is attained by allowing Coulombic friction to be present in the axle of the wheel. Under normal operation, Coulombic friction is minimized by thoroughly lubricating the contact area. If the contact area is not lubricated before collecting data, the friction will no longer be negligible.

The amplitude readings are indicated by a pointer connected to the wheel which points to a graduated scale positioned around the wheel. The scale is not graduated in any particular units, and thus can be described in a system of radians or degrees if desired. Specific units are not necessary, since the decay of the amplitude can be described generically without loss of meaning.

The time measurements are made by placing a photo gate in a position to be triggered by the pointer connected to the wheel. This photo gate is placed at the equilibrium position of the system, so all times will be half period readings as the pointer passes through this point.

Theory

The torsional oscillator is directly analogous to the much more familiar linear system consisting of a mass on a spring (as well as several other oscillating systems). The displacement angle which is measured in the rotational system is the same as the linear displacement of the mass on the spring. The moment of inertia of the wheel is analogous to the mass of the object connected to the spring. The spring constant, which is the constant of proportionality in Hooke's Law, is basically the same. The velocity dependent damping factor is also basically the same in each case. The equation of motion for a non-driven, damped harmonic oscillator is written in linear terms as:

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{1}$$

This equation is obtained from Newton's second law. m is the mass of the object connected to the spring, c is the constant of proportionality for the velocity dependent damping force, k is the spring constant, and x (which is a function of time) is the displacement from equilibrium. The dots indicate time derivatives of the x variable. For simplicity, the equilibrium point has been arbitrarily chosen to be at x = 0.

The solution of equation 1 is quite well known, and it can be found in any undergraduate level mechanics book. The solution can be obtained when two initial conditions are defined (since it is a second order differential equation). Generally, the mass is displaced from equilibrium and then released from rest. Thus, these initial conditions are:

$$x(t=0) = X_o \text{ and } \dot{x}(t=0) = 0.$$

The solution to the differential equation is now easily found to be:

$$x(t) = X_o c^{-\frac{2}{2m}t} \cos(\omega t), \tag{2}$$

where the angular frequency ω is $\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$. A decay envelope is defined by the term multiplied by the oscillating cosine term. Thus, the oscillations will decay in an exponential manner with the progress of time.

The harmonic oscillator that is being considered also has a constant Coulombic frictional force acting on it. This problem is not generally part of a typical undergraduate mechanics course, but it has been solved before. When this contact friction is the only damping force, the system can be examined from energy considerations. At any point, the harmonic oscillator will have a total energy given by:

$$Energy = T + V, \tag{3}$$

where T is the kinetic energy and V is the potential energy at the given point. The energy at any maximum amplitude A_n will simply be:

$$Energy = 0 + \tfrac{1}{2}k\left|A_n\right|^2. \tag{4}$$

T is equal to zero since the oscillator is not moving at this point. The next maximum amplitude (A_{n+1}), achieved on the other side of the equilibrium point, will have a total energy given by:

$$Energy = 0 + \frac{1}{2}k\left|A_{n+1}\right|^2. \qquad (5)$$

The distance that the oscillator traveled from any maximum amplitude to the next amplitude on the other side of the equilibrium position is $\left|A_n\right| + \left|A_{n+1}\right|$. Thus, if a constant frictional force of magnitude f is causing the oscillator to dissipate energy by doing work on the system, the work energy theorem can be written as:

$$\frac{1}{2}k\left|A_n\right|^2 + \frac{1}{2}k\left|A_{n+1}\right|^2 = f\left(\left|A_n\right| + \left|A_{n+1}\right|\right). \qquad (6)$$

where the change in energy of the system is equal to the work done on it. With a little bit of algebra, it can be shown that:

$$\left|A_n\right| - \left|A_{n+1}\right| = 2\frac{f}{k}. \qquad (7a)$$

Equation 7a says that the amplitude will decay at a constant rate, since the difference between successive amplitudes is always constant. These amplitudes are on opposite sides of equilibrium, so the derivation must be extended to successive peaks occurring on the same side of equilibrium.

In order to study peaks occurring on the same side of equilibrium, the derivation must be generalized to the possibility that the friction force acting on the system may not be the same magnitude in each direction. If f_a is the friction opposing motion in the negative direction and f_b is the friction opposing motion in the positive direction, the derivation is slightly changed. If $\left|A_n\right|$ is on the positive side of equilibrium, then equation 7a becomes:

$$\left|A_n\right| - \left|A_{n+1}\right| = \frac{2f_a}{k}. \qquad (7b)$$

Subsequently, the difference between the next two peaks will be:

$$\left|A_{n+1}\right| - \left|A_{n+2}\right| = 2\frac{f_b}{k}. \tag{7c}$$

Thus, when the plot of $|x(t)|$ is examined, the amplitude difference from $\left|A_n\right|$ to $\left|A_{n+2}\right|$ is $\frac{2}{k}\left(f_a + f_b\right)$.

This frictional force must be incorporated into the differential equation of motion of the oscillator. The frictional force opposes the motion of the oscillator, but with a constant magnitude in each direction. If this damping force were to be represented mathematically, it would appear:

$$F(t) = \begin{Bmatrix} f_a & t \in (jp, (j+\frac{1}{2})p] & j = 0,1,2,3\ldots \\ -f_b & t \in ((j+\frac{1}{2})p, (j+1)p] & j = 0,1,2,3\ldots \end{Bmatrix}.$$

The new equation of motion is now:

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{8}$$

It should be noted that the period of the square wave will be the same period as the oscillating portion of the differential solution, since it is dependent on the direction that the oscillator is moving.

This is now a very complicated problem to solve. The problem will assume the same initial conditions as mentioned earlier for the case of no Coulombic friction. These conditions were:

$$x(t=0) = X_o \text{and } \dot{x}(t=0) = 0.$$

The damping function must begin positive for the first half of the period to retard the oscillator's motion. Similarly, the damping function must be negative during the second half of the period for the same reason. Solution by the method of Laplace transforms is particularly useful for differential equations such as this one. This method is particularly good at handling discontinuous functions, and initial value problems can be solved quite nicely. The Laplace transform changes a function of one variable into a function of another variable. The Laplace transform operator is defined to be:

$$\mathrm{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)dt = f(s)$$

This transformation is useful for the solution of differential equations, because differentials can be transformed into algebraic terms. The algebraic terms are then solved and transformed back with the inverse Laplace transform[2,3,4].

If the theory of Laplace transforms is applied, and a considerable amount of algebra is performed, the solution is found to be:

$$x(t) = X_O\left(f(t) + \frac{c}{2\omega m}h(t)\right) + \frac{f_a}{k}\left(1 - f(t) - \frac{c}{2\omega m}h(t)\right) \tag{9}$$
$$+\frac{2(f_a + f_b)}{kp}\sum_{n=1}^{\infty}(-1)^n\left(1 - f(t-\tfrac{np}{2}) - \frac{c}{2\omega m}h(t-\tfrac{np}{2})\right)\mathrm{S}_{\frac{np}{2}}(t)$$

where $f(t) = e^{-\frac{c}{2m}t}\cos(\omega t)$, $h(t) = e^{-\frac{c}{2m}t}\sin(\omega t)$, $\mathrm{S}_{\frac{np}{2}}(t) = \left\{\begin{matrix} 1 & t \geq \frac{np}{2} \\ 0 & t < \frac{np}{2} \end{matrix}\right\}$,

$p = \frac{2\pi}{\omega}$, and $\omega = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$. It is very interesting to note that the period of oscillation for this situation where Coulombic friction is present is exactly the same as when there is no contact friction. In other words, the existence of friction in the axle of the system does not change the period of the oscillations. It is important to know that this is true of the peaks only. An examination of the periodicity of the crossing points (the times when the mass crosses equilibrium) reveals that it is not constant. This is important for scientists using timing devices which rely on oscillating systems that may have a constant frictional damping force present. The peaks will occur with constant periodicity, while the equilibrium points will not.

For the purpose of this experiment, only the peak amplitudes are desired. The solution above is the full oscillating solution, so the peak amplitudes must be extracted. If integral multiples of the period are chosen, then the peak amplitudes will be the only values given by the solution. So, making the substitution t->jp in equation 9, the solution takes the discrete form:

$$x(jp) = X_o\left(e^{-\frac{c}{2m}(jp)}\right) + \frac{f_a}{k}\left(1 - e^{-\frac{c}{2m}(jp)}\right) \tag{10}$$
$$+\frac{2(f_a + f_b)}{kp}\sum_{n=1}^{\infty}(-1)^n\left(1 - e^{-\frac{c}{4m}(2j-n)p}(-1)^n\right).$$

Equation 10 reduces to the appropriate form in each of the limiting cases. When the Coulombic damping force is not present, when $f_a = f_b = 0$, equation 10 reduces to:

$$x(jp) = X_o c^{-\frac{2}{2m}(jp)}, \tag{11}$$

which is instantly recognizable as the exponential decay envelope expected (eq. 2). If the velocity dependent damping force is removed by setting c equal to 0, then equation 10 reduces to:

$$x(jp) = X_o - \frac{2(f_a + f_b)}{kp}(jp). \tag{12}$$

This is the equation of a line with a slope of $-\frac{2(f_a + f_b)}{kp}$. This agrees with the linear decay envelope derived from energy considerations earlier.

The decay envelope of the peaks on the other side of equilibrium should be the same as that of the positive side. Indeed, when the substitution t->$(\frac{p}{2} + jp)$ is made, the decay envelope is exactly as intuition would predict. The absence of Coulombic friction results in an envelope that is the negative of the positive side's exponential form. Similarly, in the case of no velocity damping, the envelope is a straight line with a slope $+\frac{2(f_a + f_b)}{kp}$.

To verify further that the solution to the differential equation is correct, it was compared to a numerical solution. The agreement between the two solutions is nearly perfect (fig. 1).

Experimental Procedure

The procedure for collecting data from this system is quite simple. The wheel was displaced from equilibrium to some amplitude, then it was allowed to oscillate freely. The amplitude measurements were simply read at each maximum displacement attained. Corresponding time measurements were made by the photo gate. The times were then averaged to be used as the period for these oscillations. This averaging is justified by the fact that the peaks occur with constant periodicity. The trials were repeated with different currents sent through the coils of the electromagnet (ranging from 0 Amperes to 0.8 Amperes).

This effect requires careful experimental design to measure on this particular apparatus. The photo gate had to be placed at the equilibrium point for obvious reasons. This point ensured that the oscillator's pointer would pass through the gate for large amplitudes as well as small amplitudes. It also ensured that timing errors would be minimized, since the equilibrium point is the position where the oscillator has the most kinetic energy. When the pointer moves through the photo gate too slowly, the timer is triggered by both the leading and trailing edges of the pointer, rather that one single trigger by the pointer itself. So, the time measured is the time it takes to travel from the equilibrium point to the maximum amplitude and then back to the equilibrium point.

Experimental Results and Discussion

The results obtained from the torsional oscillator, at a first glance, illustrate the effects of the two forces on the system quite well. The data collected from the system when no current is sent to the electromagnet corresponds exactly with the theoretical expectation for a system decaying under the influence of a constant damping force (fig. 2).

The slopes of the decay envelope on both the positive and the negative sides of equilibrium were obtained by fitting a linear model to the experimental data (table 1).

Added current through the electromagnet displays the exponential nature of the decay becoming much more prominent (figs. 3 - 6). The lower currents result in a more difficult situation, where the Coulombic frictional force and the electromagnetic velocity dependent damping force are comparable in magnitude and neither dominates the motion.

Fitting the New Model

There are two parameters that can be varied when fitting the model to the experimental data, and there are two parameters that can be determined experimentally. The value of X_0 is obviously the initial amplitude from which the pendulum is allowed to oscillate. The value for the multiplier of the summation term, $\frac{(f_a + f_b)}{k}$, is obtained from the slope of the plot which has no velocity damping present. With these two values fixed, a least squares method could be used to fit the model to the experimental data. $\frac{c}{2m}$ and $\frac{f_a}{k}$ could both be minimized in the fit.

The fit was accomplished by making the discrete model continuous. A substitution was made such that jp->t. This method of "smoothing" the function is not totally successful, due to the complicated nature of the model, but it is sufficient for a fitting program to minimize the parameters.

When the parameters have been minimized, the values are inserted back into the discrete function to generate values for the peak amplitudes. Plotting the model's values superimposed on the plot of the corresponding experimental data allows for a visual examination of the fit. All of the fits are very good (figs. 7 - 10).

Success of the Model

The model turns out to be quite successful in fitting the experimental data. The results of the parameter minimization are shown in table 2. The most convincing argument that the model is fitting properly is the linear relationship between the $\frac{c}{2m}$ values obtained and the corresponding currents used in the electromagnet (fig. 11). Since the apparatus was designed so the magnitude of the velocity damping force (c in the equation of motion) would be linearly dependent on the current through the coils of the electromagnet. By finding this same relationship by least squares fit, the model is verifying that it is fitting properly.

The most obvious result from the parameter minimization is the fact that the values obtained for $\frac{f_a}{k}$ are not constant from one trial to another. According the differential equation of motion, the magnitude of f_a is invariant, since it is a constant inherent in the system. This assumption may not be correct. The amount of Coulombic friction in the system could be a dynamic quantity. If the apparatus is allowed to sit untouched for a length of time, the contact areas in the axle will be altered by dust and other particles which can settle. The coefficient of the friction may also change with time as the lubricant evaporates. For these reasons, the assumption that a particular trial may have a constant amount of Coulombic friction is a good approximation, but the assumption that the magnitude of the frictional force remains constant from one trial to another is very likely wrong. This is especially true for trials performed on different days. For this reason, differing values of the friction term are quite justified. The fit could possibly be made more meaningful if both the f_a and f_b terms are made variable parameters.

Conclusion

The analysis and solution of this system leads to some very nice conclusions. First of all, the solution to the differential equation is correct, since it agrees with the numerical solution with such a high degree of accuracy. When this solution is fitted with the experimental data, the agreement is also excellent. This implies that the differential equation of motion contains accurate representations of the forces present in the system. The step function used to represent the Coulombic friction is a novel portrayal of the actual force, but it may not be totally accurate. The nature of the friction occurring in the axle implies that the magnitude of the force is more complex than this model accounts for. It is quite likely that a "sticking" and "slipping" effect is occurring, where the adhesion is occurring between the surfaces when below a certain velocity. When the mass attains enough kinetic energy, it can escape this adhesive force and slide over the surface with no sticking affect. The velocity of the pendulum is constantly increasing and decreasing, so when it is moving slower near the peak amplitudes, this effect is most likely coming into play. The magnitude of the friction force thus increases near the peaks and more energy is dissipated than when the pendulum swings through the equilibrium point. It is in this direction that research of this system will undoubtedly go, as the more complex form of Coulombic friction is introduced into the model. Analytic solution of the new differential equation may be impossible, but numerical solution probably will be possible.

The analytic model of this system does have some practical usefulness. In mechanical engineering, the model could be used to asses the magnitude of the Coulombic friction present if the velocity dependent damping constant is known. It could also be used in the design of various mechanical oscillators which will most certainly have contact friction inherent in the motion. The design of timing mechanisms could benefit from the knowledge that the amplitude peaks retain the same periodicity despite the presence of contact friction.

As the project stands, the usefulness is almost entirely pedagogical. Mechanical engineers and physicists have known about this type of friction, and the modeling has probably been done numerically. An analytic treatment has never been included in any textbooks, most likely due to the prohibitively difficult nature of the solution. This solution process offers an important method of solving differential equations by Laplace transforms, as well as providing some interesting information about a system with these particular damping forces.

Tables

Table 1 (Slopes of linear decay envelopes)

Trial	Slope (Absolute Value)
Positive Side of Equilibrium	0.377±0.002
Negative Side of Equilibrium	0.345±0.002

Table 2 (Values obtained for variable parameters)

Current through electromagnet (Amperes)	$\frac{c}{2m}$	$\frac{f_a}{k}$
0.25	0.0293	2.49
0.40	0.112	0.808
0.60	0.258	0.294
0.80	0.483	0.341

Figure 1
Model with Numerical Solution

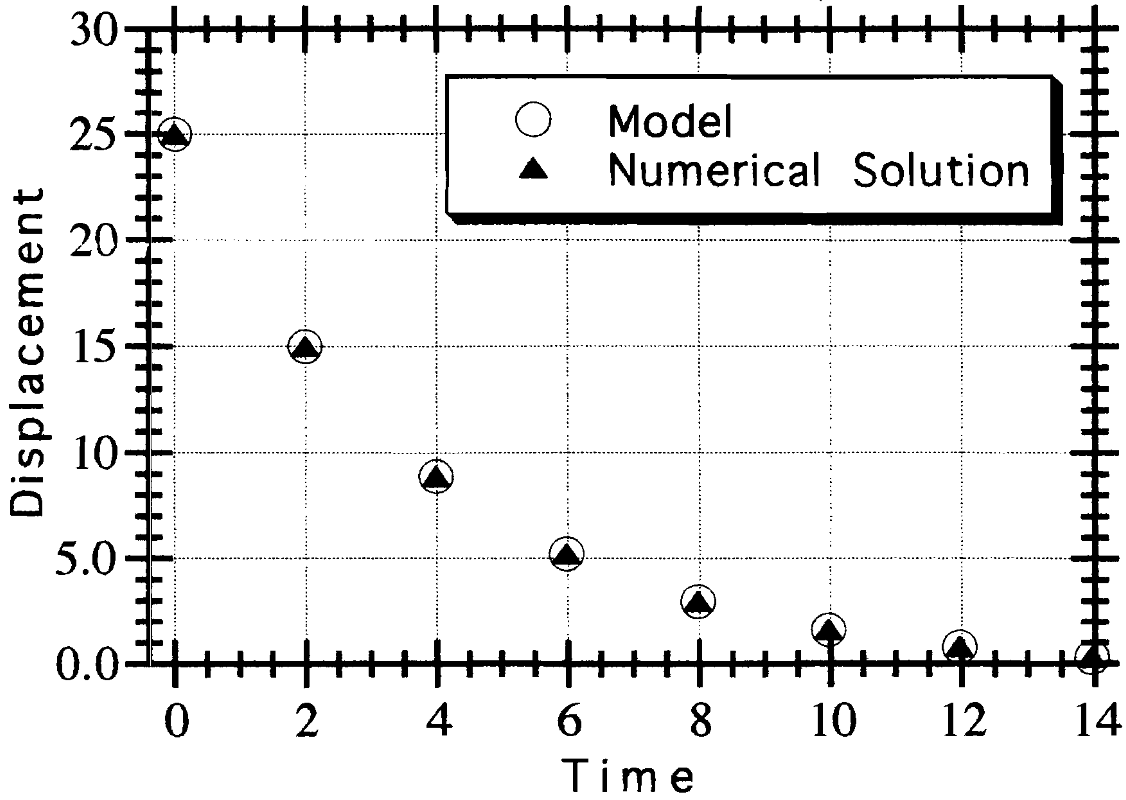


Figure 2
Experimental Data (I = 0 A)

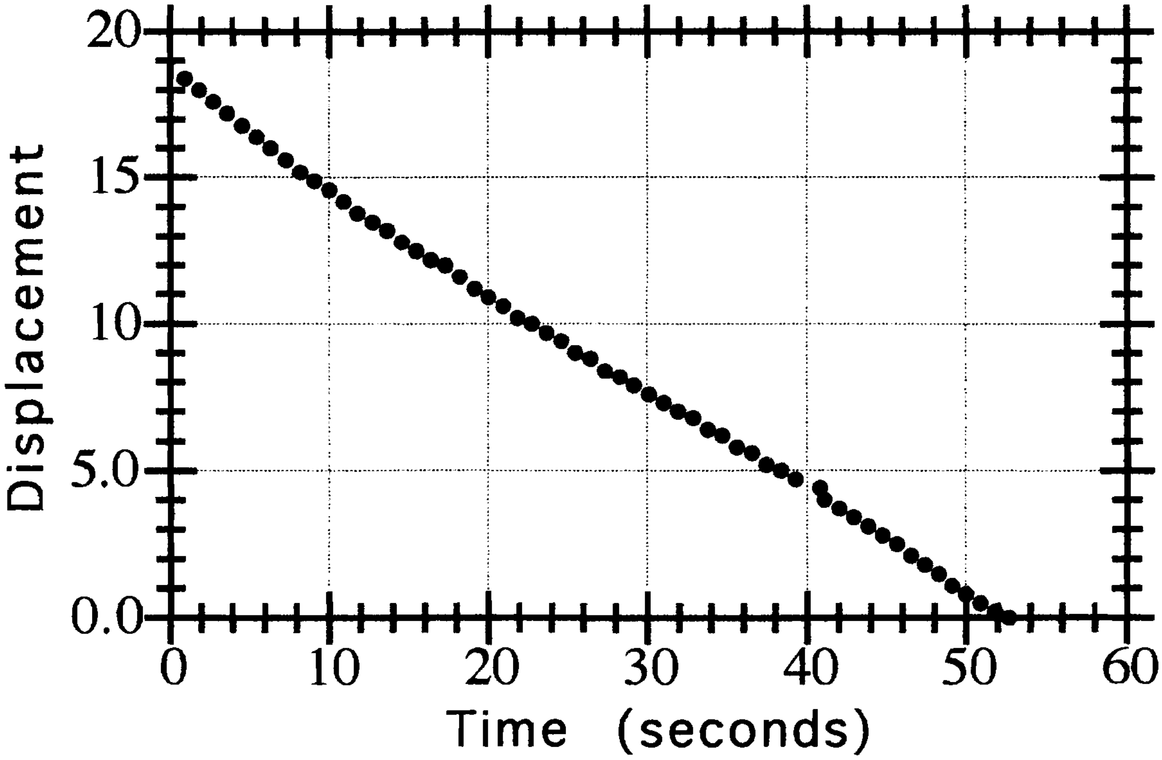


Figure 3
Experimental Data (I = 0.25 A)

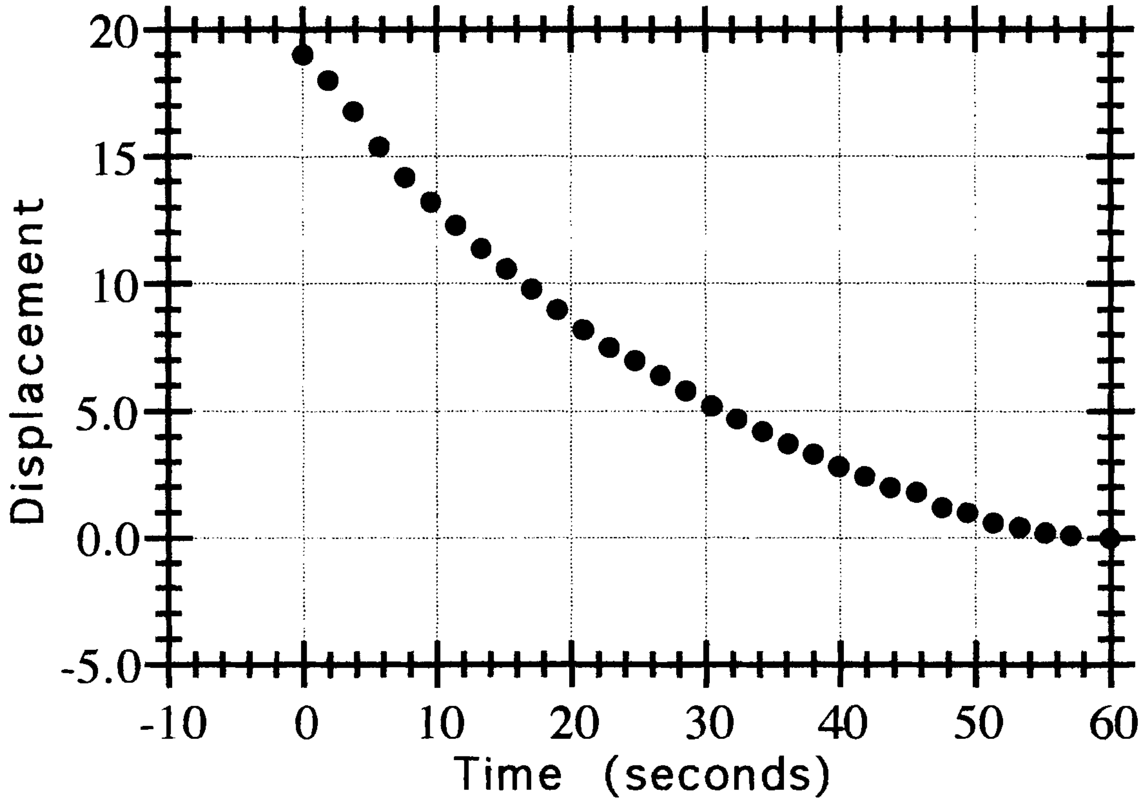


Figure 4
Experimental Data (I = 0.4 A)

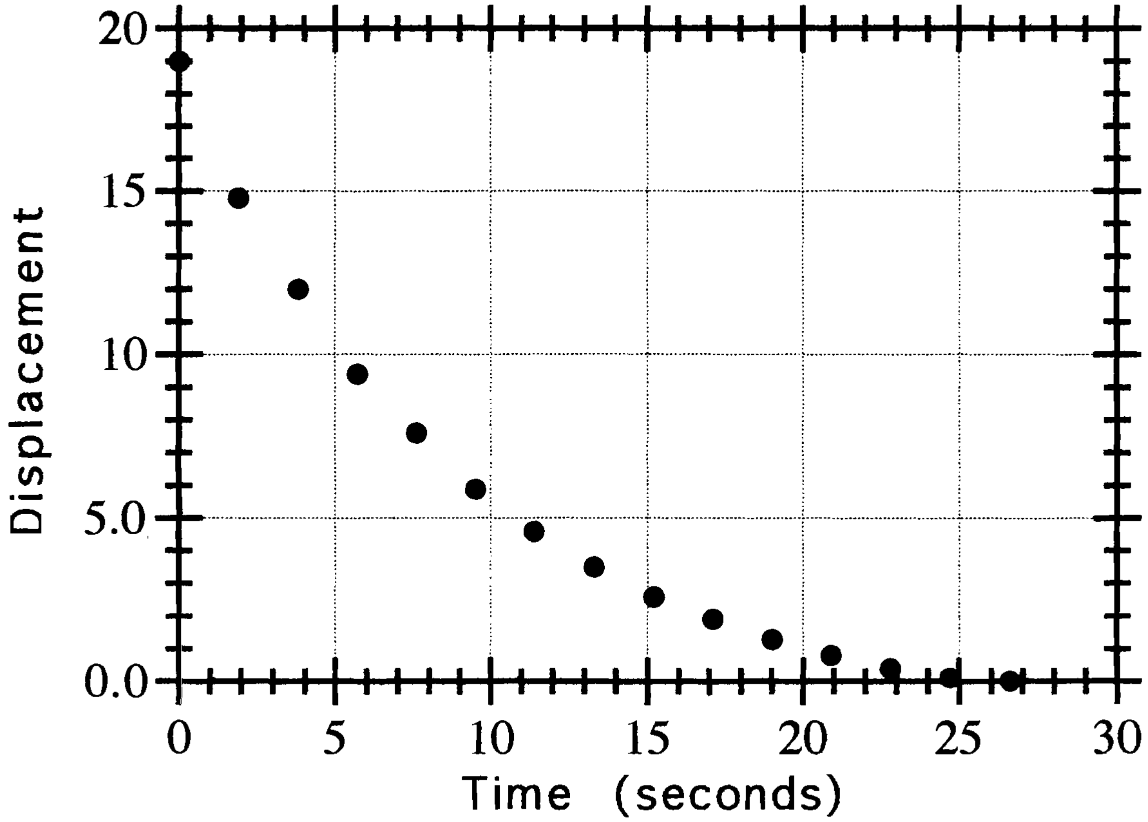


Figure 5
Experimental Data (I = 0.6 A)

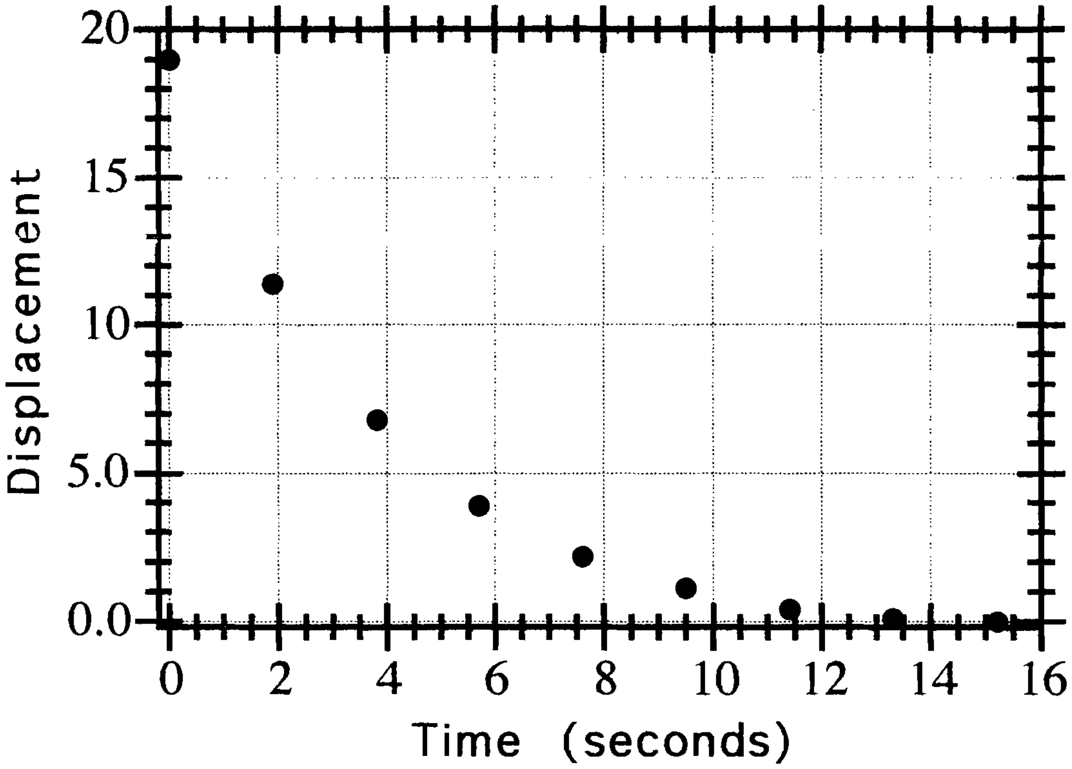


Figure 6
Experimental Data (I = 0.8 A)

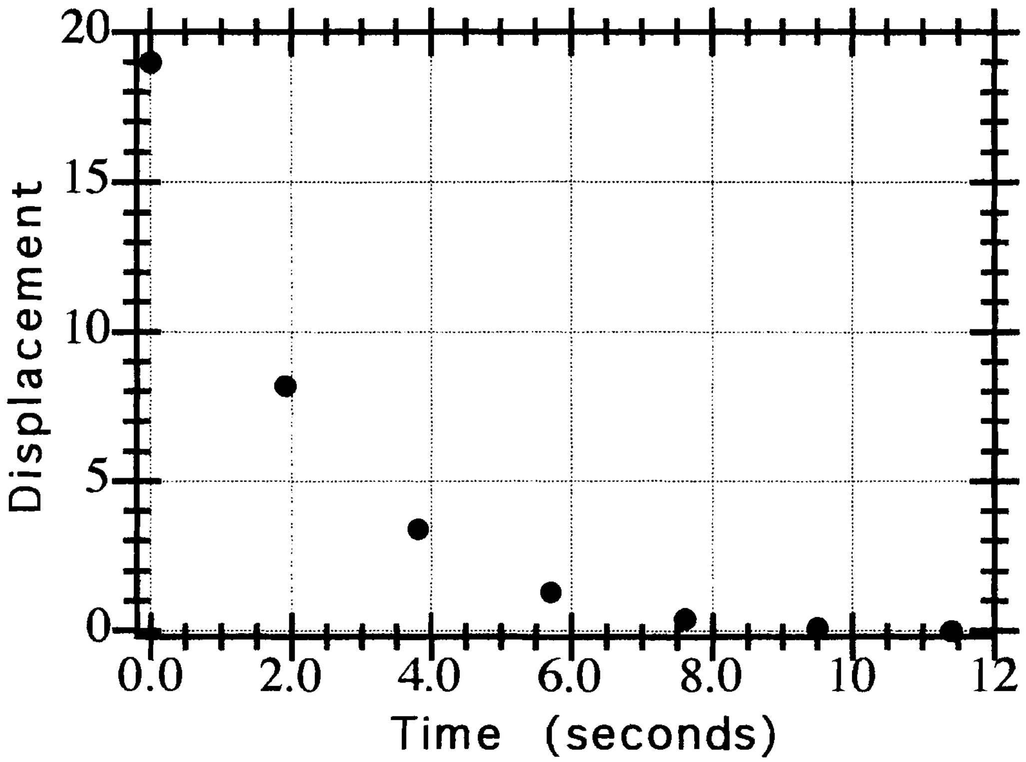


Figure 7
Model's Fit (I = 0.25 A)

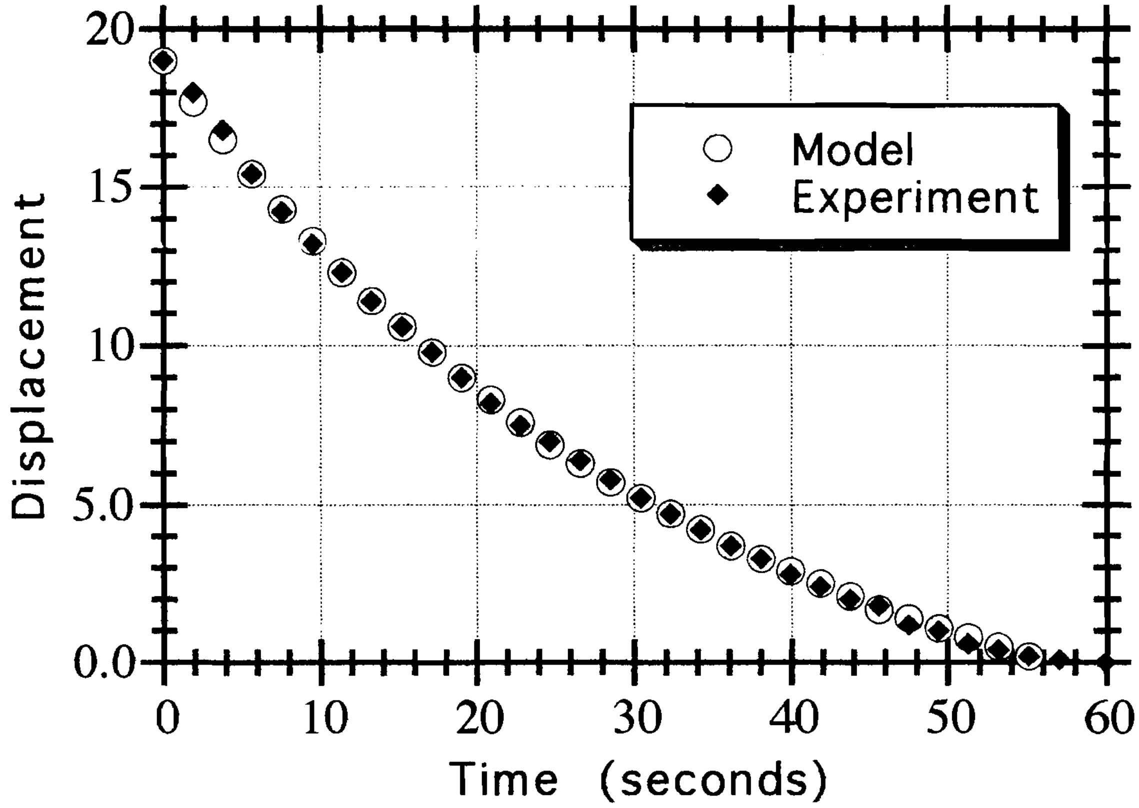


Figure 8
Model's Fit (I = 0.4 A)

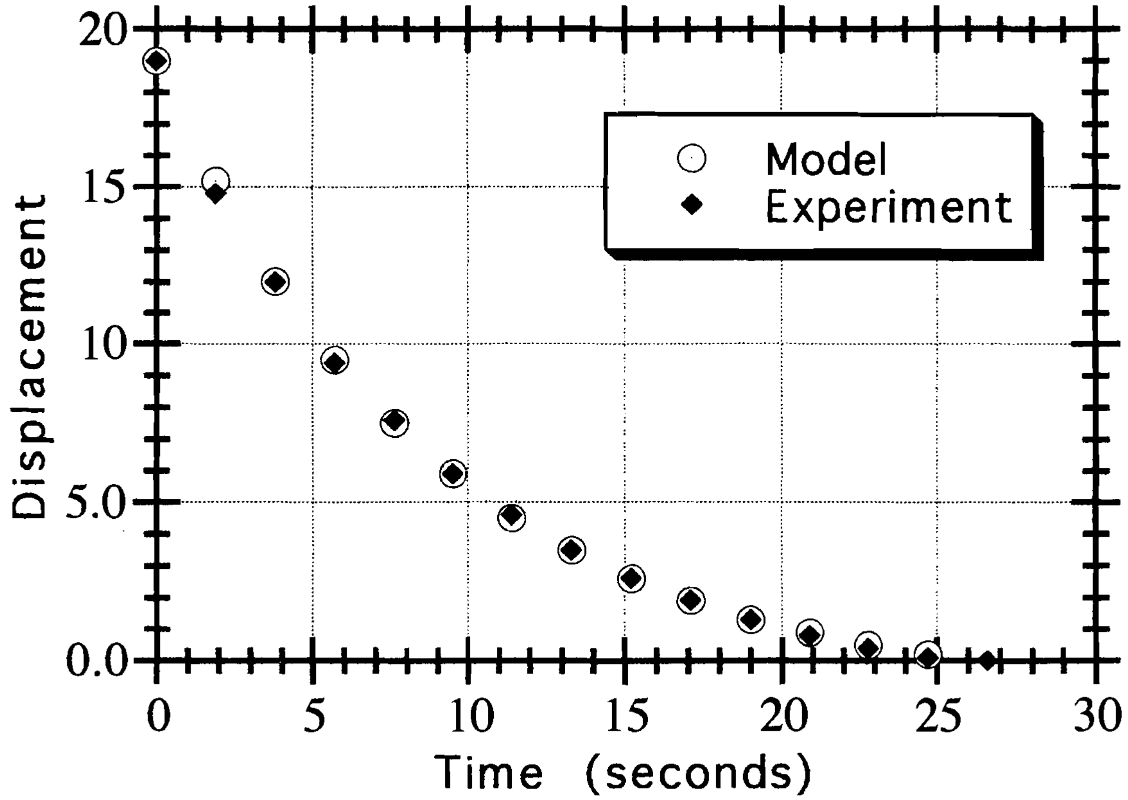


Figure 9
Model's Fit (I = 0.6 A)

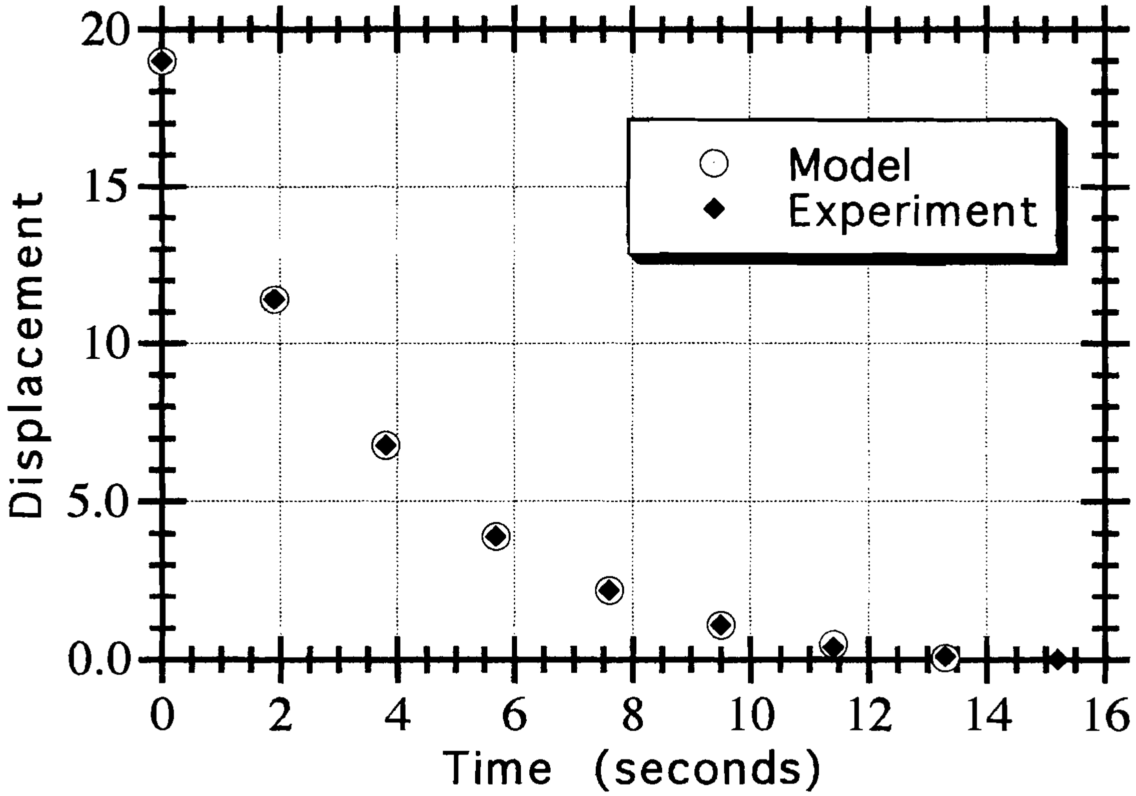


Figure 10
Model's Fit (I = 0.8 A)

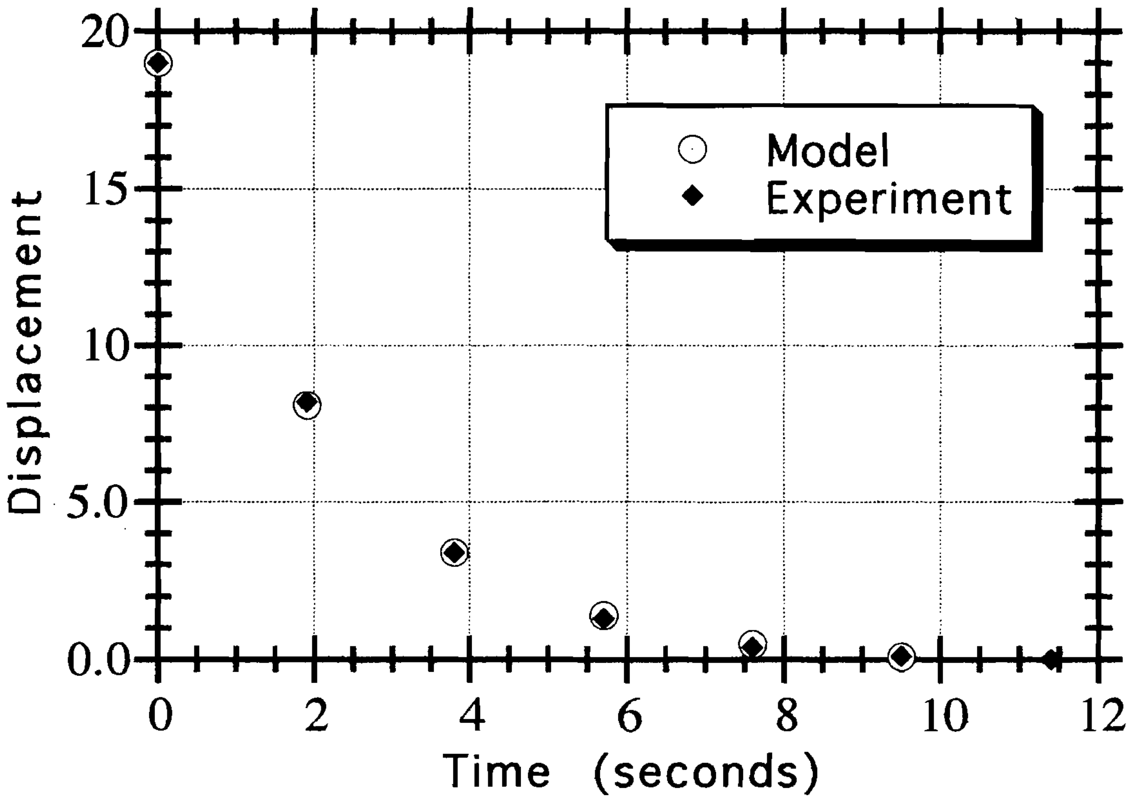
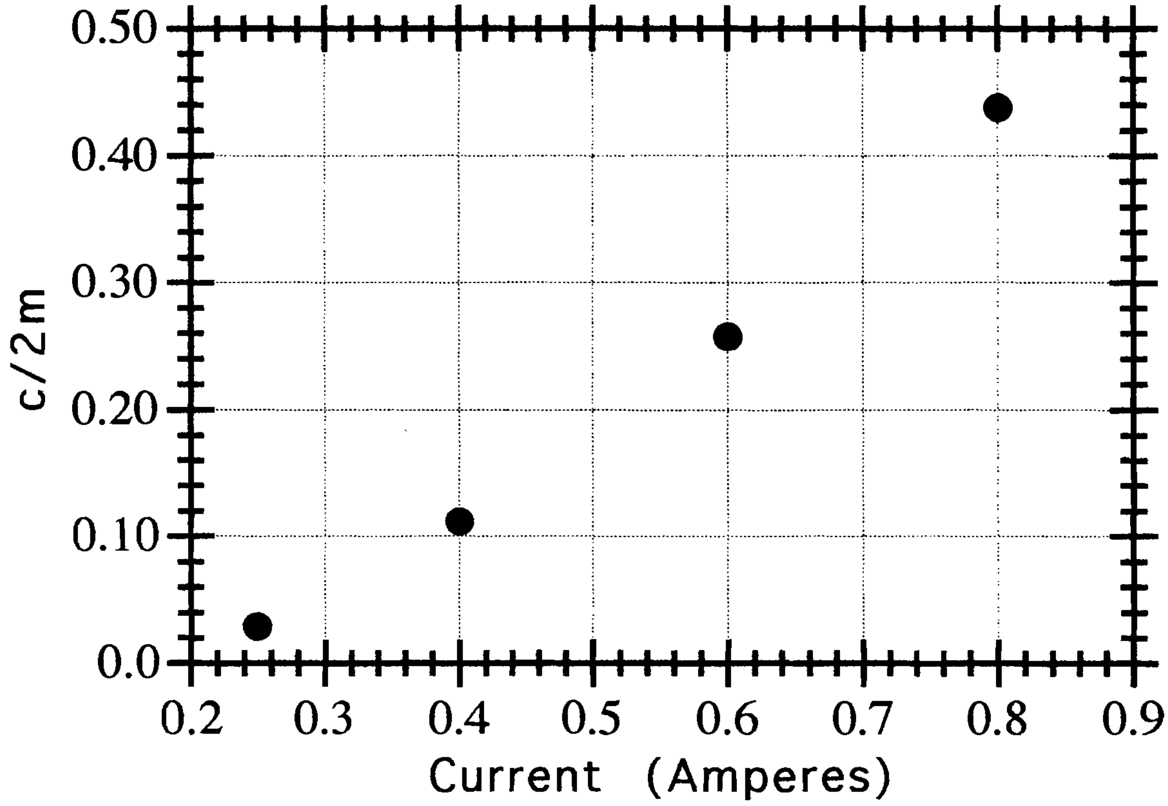


Figure 11
c/2m vs. Current



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