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Two Rosa-type Labelings of Uniform k-distant Trees and a New Class of Trees

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Abstract

A *k-distant tree* consists of a main path, called the *spine*, such that each vertex on the spine is joined by an edge to an end-vertex of at most one path on at most k vertices. Those paths, along with the edge joining them to the spine, are called *tails*. When every vertex on the spine has exactly one incident tail of length k we call the tree a *uniform* k-distant tree. We show that every uniform k-distant tree admits both a graceful- and an α-labeling.

For a graph G and a positive integer a, define $app_a(G)$ to be the graph obtained from appending a leaves to each leaf in G. When G is a uniform k-distant tree, we show that $app_a(G)$ admits both a graceful- and an α-labeling.

1 Introduction

Let G be a graph. Denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. A *k-distant tree* consists of a main path, called the *spine*, such that each vertex on the spine is joined by an edge to an end-vertex of at most one path on at most k vertices. Those paths, along with the edge joining them to the spine, are called *tails*. When every vertex on the spine has exactly one incident tail of length k we call the tree a *uniform* k-distant tree. [1]

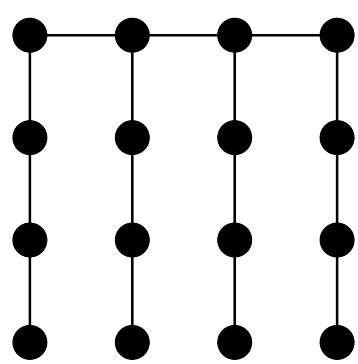

Figure 1: An example of a uniform 3-distant tree.

A *graceful labeling* of a graph G on n vertices is a one-to-one function from the vertices of G to the set $\{0, ..., |E(G)|\}$ such that the induced edge labels given by $|f(u) - f(v)|$, for every $uv \in E(G)$, are all distinct. If a graph admits a graceful labeling then that graph is said to be *graceful*. Graceful labelings were first introduced by Rosa in 1967 [10] to generate graph decompositions. As a historical note, Rosa used the term β-valuation for what is now commonly known as a graceful labeling. The term "graceful labeling" was coined years later.

[1] Fundamental definitions can be found in the appendix.

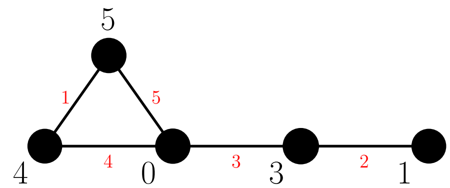

Figure 2: An example of a graceful labeling of a graph. The induced edge labels are displayed in red.

Let G and H be graphs. A *G-decomposition of H* is a set of edge-disjoint subgraphs of H, $\mathscr{G} = \{G_1, G_2, \ldots, G_t\}$ such that $\cup_{i=1}^{t} G_i = E(H)$ and for each $1 \leq i \leq t$, $G_i \cong G$. Identify the vertices of K_n with the elements of $\mathbb{Z}_n$. Now, consider a G-decomposition of K_n, $\mathscr{D}$. If applying the permutation $(0, 1, 2, \ldots, n-1)$ to the vertices of the elements in $\mathscr{D}$ is an automorphism of $\mathscr{D}$ then we say that $\mathscr{D}$ is *cyclic*. In 1963 Ringel conjectured that for every tree T on n edges there exists a T-decomposition of K_{2n+1}, the complete graph on $2n+1$ vertices. It was further conjectured by Kötzig that not only does K_{2n+1} admit a T-decomposition, but it also admits a cyclic T-decomposition. The conjecture that for every tree, T, on n edges there exists a cyclic T-decomposition of K_{2n+1} is known as the Ringel-Kötzig conjecture. The connection between graph labelings and cyclic graph decompositions is as shown in the following theorem.

Theorem 1.1 (Rosa [10])**.** *Let G be a graph with n edges. If G is graceful then a cyclic G-decomposition of K_{2n+1} exists.*

Combining Theorem 1.1 with the Ringel-Kötzig conjecture yields the following famous conjecture, which has motivated many of the results on graceful labelings.

Conjecture 1.2 (Graceful Tree Conjecture)**.** *All trees are graceful.*

For a given tree, Theorem 1.1 provides one graph decomposition. In order to strengthen this result, we must visit another labeling that was defined in Rosa's original paper. For a graph G, we call a function $f : V(G) \to \{0, \ldots, |E(G)|\}$ an α-labeling of G if f is a graceful labeling of G with the additional property that there exists some integer λ such that for all $uv \in E(G)$, $f(u) \leq \lambda$ and $f(v) > \lambda$. Notice that if G admits an α-labeling, then G is necessarily bipartite. The connection between α-labelings and cyclic graph decompositions is as shown below. Note that α-labelings lead to an infinite number of graph decompositions.

Theorem 1.3 (Rosa [10])**.** *Let G be a graph with n edges. If G admits an α-labeling then for every positive integer x there exists a cyclic G-decomposition of K_{2nx+1}.*

While graph labelings continue to be used in the study of graph decompositions, they have since become a thoroughly studied subject of their own. A dynamic survey [5] is maintained by Gallian and contains over 1400 references. Thus far, not many general results have been obtained in order to attack the Graceful Tree Conjecture. It is common practice to show that various classes of trees are graceful. To this end, we define the following classes of trees. A *caterpillar* is a tree of order three or more in which the removal of its leaves produces a path.

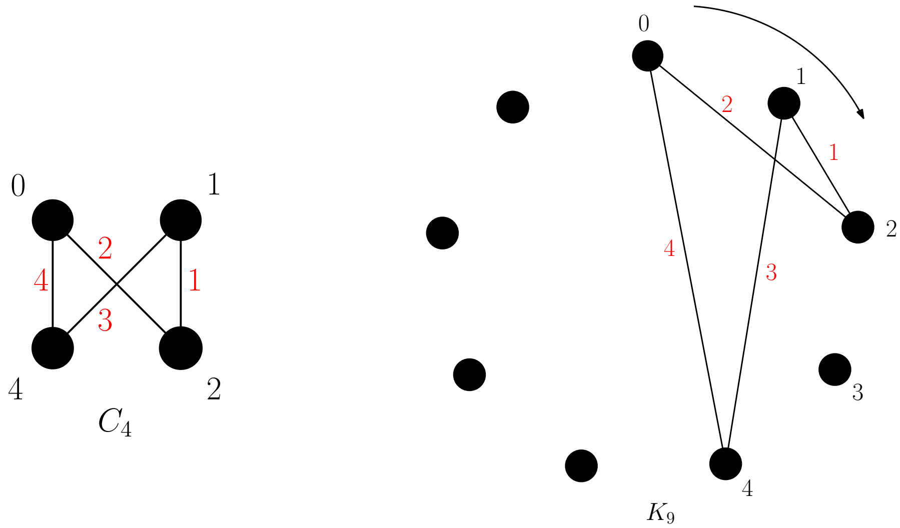

Figure 3: A graceful labeling of C_4 (left), along with the corresponding embedding of C_4 into K_9 that generates a cyclic decomposition. Notice that, in this case, the graceful labeling is also an α-labeling.

A *lobster* is a tree with the property that the removal of its leaves produces a caterpillar. We note that uniform 1-distant trees are special caterpillars and uniform 2-distant trees are special lobsters. It is known that all caterpillars are graceful [10], and that all lobsters with a perfect matching are graceful [7], yet it remains an open problem to determine if all lobsters are graceful. Murugan showed in [9] that, among admitting other labelings, all uniform k-distant trees admit a graceful labeling. Aldred and Mckay [2] have shown that all trees on at most 27 vertices are graceful. Fang [3] has extended this to include all trees on at most 35 vertices.

In 1980, Graham and Sloane defined the following graph labeling as a means of obtaining additive bases of sets of integers. A *harmonious labeling* of a graph G is a function $f : V(G) \to \{0, ..., |E(G)|-1\}$ such that f is injective, and the induced edge labels given by $(f(u)+f(v)) \pmod{m}$, for every $uv \in E(G)$, are all distinct. In the case where G is a tree, exactly one vertex label can be repeated exactly once.

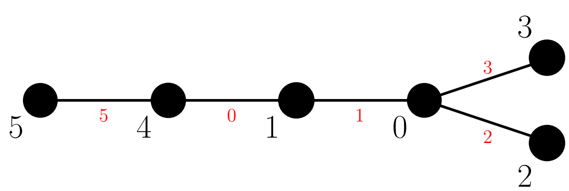

Figure 4: An example of a harmonious labeling of a graph.

An analogous version of the Graceful Tree Conjecture has been formulated for harmonious labelings.

Conjecture 1.4 (Graham and Sloan [6], 1980)**.** *All trees are harmonious.*

Some results have been obtained the positively support this conjecture. Aldred and Mckay [2] have shown that all trees on at most 26 vertices are harmonious, and this was extended to all trees on at most 31 vertices by Fang [4]. Graham and Sloane showed in [6] that all caterpillars are harmonious. It is not known if all lobsters are harmonious. In [1], it is shown that every uniform k-distant tree with an even number of vertices admits a harmonious labeling. These results were extended in [8] to include all uniform k-distant trees as harmonious graphs. Our current research is motivated by the previous two results on uniform k-distant trees, along with the Graceful Tree conjecture.

2 Uniform k-distant Trees

2.1 Graceful Labeling

In this section we produce a function which is a graceful labeling of a uniform k-distant tree. To define the function, it is necessary for us to name the vertices of the uniform k-distant tree in a specific way. Name the vertices as follows.

1. Arrange the vertices of the tree so that the tails extend downward from the spine.
2. To begin, label the vertex in the upper left-hand corner v_0. Moving down the tail, label the next vertex v_1, the next v_2, etc., until the end of the tail is reached, i.e. v_k.
3. Continue naming the vertex directly to the right (on the next tail) with v_{k+1}. Proceed in the same way up the tail toward the spine.
4. After reaching the spine, or the vertex v_{2k+1}, move to the vertex directly to the right (on the spine), and continue down the next tail.
5. Proceed in this way until all vertices are named. The figure below exemplifies the procedure.

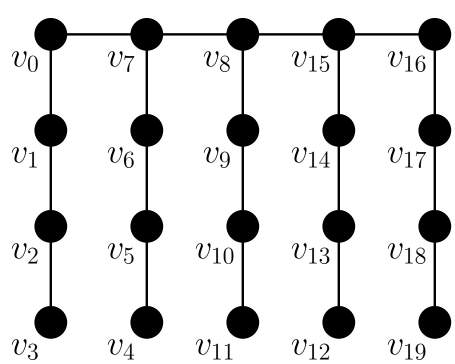

Figure 5: Naming the vertices of a uniform 3-distant tree.

To assign labels to the vertices, we define a function f from the n vertices of the graph to the set $\{0, 1, ..., n-1\}$, where $n-1$ is the number of edges.

$$f(v_i) = \begin{cases} \dfrac{i}{2} & \text{if } i \text{ is even} \\ n - \dfrac{i+1}{2} & \text{if } i \text{ is odd} \end{cases}$$

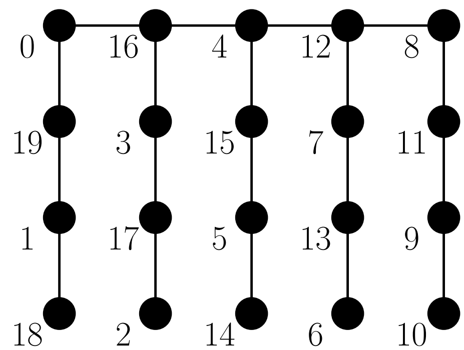

Figure 6: A graceful labeling of the vertices of a uniform 3-distant tree.

We will now show that given a uniform k-distant tree, the function described above is a graceful labeling of that tree.

Theorem 2.1. *Every uniform k-distant tree is graceful.*

Proof. Let G be a uniform k-distant tree, where k is the number of edges in each tail, and let s denote the number of vertices in the spine. Note that $n = s(k+1)$, where n is the number of vertices of G. Let $V(G)$ be the set of vertices of G and $E(G)$ be the set of edges of G.

Let $f : V(G) \to \{0, 1, ..., n-1\}$, where $n-1$ is the number of edges of G, be defined as in the algorithm above. To prove that G is graceful, we must show that the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(G)$ are all distinct.

We can view $E(G)$ as the union of three sets of edges: edges in the tails, edges between consecutively named vertices on the spine (i.e. $v_{2k+1}v_{2k+2}$), and edges between non-consecutively named vertices on the spine. Let T denote the set of edges in the tails, I denote the set of edges between consectutively named vertices on the spine, and H denote the set of edges between non-consecutively named vertices on the spine. So, $E(G) = T \cup I \cup H$.

Let us examine each set making up $E(G)$ more closely. T is the set of edges in the tails. Thus,

$$\begin{aligned} T = &\left\{v_0v_1, v_1v_2, ..., v_{k-1}v_k\right\} \cup \left\{v_{k+1}v_{k+2}, v_{k+2}v_{k+3}, ..., v_{2k}v_{2k+1}\right\} \\ &\cup \left\{v_{2k+2}v_{2k+3}, v_{2k+3}v_{2k+4}, ..., v_{3k+1}v_{3k+2}\right\} \cup \left\{v_{3k+3}v_{3k+4}, v_{3k+4}v_{3k+5}, ..., v_{4k+2}v_{4k+3}\right\} \\ &\cup ... \cup \left\{v_{(s-1)(k+1)}v_{(s-1)(k+1)+1}, v_{(s-1)(k+1)+1}v_{(s-1)(k+1)+2}, ..., v_{s(k+1)-2}v_{s(k+1)-1}\right\}. \end{aligned}$$

Let T_0 be the set of induced edge labels given by $|f(u) - f(v)|$ for all $uv \in T$. Here we examine the four cases of T_0.

Case 1: Both k and s are even.

$$\begin{aligned} T_0 = \{&|0 - [s(k+1) - 1]|, ..., |[s(k+1) - \tfrac{k}{2}] - \tfrac{k}{2}|\} \cup \{|[s(k+1) - \tfrac{k+2}{2}] - \tfrac{k+2}{2}|,\\ &..., |\tfrac{2k}{2} - [s(k+1) - \tfrac{2k+2}{2}]|\} \cup \{|\tfrac{2k+2}{2} - [s(k+1) - \tfrac{2k+4}{2}]|, ...,\\ &|[s(k+1) - \tfrac{3k+2}{2}] - \tfrac{3k+2}{2}|\} \cup ... \cup \{|[s(k+1) - \tfrac{(s-1)(k+1)+1}{2}] - \tfrac{(s-1)(k+1)+1}{2}|,\\ &..., |\tfrac{s(k+1)-2}{2} - [s(k+1) - \tfrac{s(k+1)}{2}]|\}\\ = \{&|1 - s(k+1)|, ..., |s(k+1) - k|\} \cup \{|s(k+1) - (k+2)|, ...,\\ &|(2k+1) - s(k+1)|\} \cup \{|(2k+3) - s(k+1)|, ..., |s(k+1) - (3k+2)|\} \cup ... \cup \{|k|, ..., |-1|\}. \end{aligned}$$

Case 2: k is even, s is odd.

$$\begin{aligned} T_0 = \{&|0 - [s(k+1) - 1]|, ..., |[s(k+1) - \tfrac{k}{2}] - \tfrac{k}{2}|\} \cup \{|[s(k+1) - \tfrac{k+2}{2}] - \tfrac{k+2}{2}|,\\ &..., |\tfrac{2k}{2} - [s(k+1) - \tfrac{2k+2}{2}]|\} \cup \{|[s(k+1) - \tfrac{2k+4}{2}] - \tfrac{2k+4}{2}|, ...,\\ &|[s(k+1) - \tfrac{3k+2}{2}] - \tfrac{3k+2}{2}|\} \cup ... \cup \{\tfrac{(s-1)(k+1)}{2} - [s(k+1) - \tfrac{(s-1)(k+1)+2}{2}]|,\\ &..., |[s(k+1) - \tfrac{s(k+1)-1}{2}|\}\\ = \{&|1 - s(k+1)|, ..., |s(k+1) - k|\} \cup \{|s(k+1) - (k+2)|, ...,\\ &|(2k+1) - s(k+1)|\} \cup \{|(2k+3) - s(k+1)|, ..., |s(k+1) - (3k+2)|\} \cup ... \cup \{|-k|, ..., |1|\}. \end{aligned}$$

Case 3: Both k and s are odd.

$$\begin{aligned} T_0 = \{&|0 - [s(k+1) - 1]|, ..., |\tfrac{k-1}{2} - [s(k+1) - \tfrac{k+1}{2}]|\} \cup \{|\tfrac{k+1}{2} - [s(k+1) - \tfrac{k+3}{2}]|,\\ &..., |\tfrac{2k}{2} - [s(k+1) - \tfrac{2k+2}{2}]|\} \cup \{|\tfrac{2k+2}{2} - [s(k+1) - \tfrac{2k+4}{2}]|, ...,\\ &|\tfrac{3k+1}{2} - [s(k+1) - \tfrac{3k+3}{2}]|\} \cup ... \cup \{|\tfrac{(s-1)(k+1)}{2} - [s(k+1) - \tfrac{(s-1)(k+1)+2}{2}]|,\\ &..., |\tfrac{s(k+1)-2}{2} - [s(k+1) - \tfrac{s(k+1)}{2}]|\}\\ = \{&|1 - s(k+1)|, ..., |k - s(k+1)|\} \cup \{|(k+2) - s(k+1)|, ...,\\ &|(2k+1) - s(k+1)|\} \cup \{|(2k+3) - s(k+1)|, ..., |(3k+2) - s(k+1)|\} \cup ... \cup \{|-k|, ..., |-1|\}. \end{aligned}$$

Case 4: k is odd, s is even.

$$\begin{aligned} T_0 = \{&|0 - [s(k+1) - 1]|, ..., |\tfrac{k-1}{2} - [s(k+1) - \tfrac{k+1}{2}]|\} \cup \{|\tfrac{k+1}{2} - [s(k+1) - \tfrac{k+3}{2}]|,\\ &..., |\tfrac{2k}{2} - [s(k+1) - \tfrac{2k+2}{2}]|\} \cup \{|\tfrac{2k+2}{2} - [s(k+1) - \tfrac{2k+4}{2}]|, ...,\\ &|\tfrac{3k+1}{2} - [s(k+1) - \tfrac{3k+3}{2}]|\} \cup ... \cup \{|\tfrac{(s-1)(k+1)}{2} - [s(k+1) - \tfrac{(s-1)(k+1)+2}{2}]|,\\ &..., |\tfrac{s(k+1)-2}{2} - [s(k+1) - \tfrac{s(k+1)}{2}]|\}\\ = \{&|1 - s(k+1)|, ..., |k - s(k+1)|\} \cup \{|(k+2) - s(k+1)|, ...,\\ &|(2k+1) - s(k+1)|\} \cup \{|(2k+3) - s(k+1)|, ..., |(3k+2) - s(k+1)|\} \cup ... \cup \{|-k|, ..., |-1|\}. \end{aligned}$$

Notice that upon further simplification, each case of T_0 results in the same set. So, for any k and any s,

$$\begin{aligned} T_0 &= \{s(k+1) - 1, ..., s(k+1) - k\} \cup \{s(k+1) - (k+2), ..., s(k+1) - (2k+1)\}\\ &\quad \cup \{s(k+1) - (2k+3), ..., s(k+1) - (3k+2)\} \cup ... \cup \{k, ..., 1\}\\ &= \{s(k+1) - i \mid i = 1, ..., k\} \cup \{s(k+1) - i \mid i = k+2, ..., 2k+1\} \cup \{s(k+1) - i \mid\\ &\quad i = 2k+3, ..., 3k+2\} \cup ... \cup \{s(k+1) - i \mid i = s(k+1) - k, ..., s(k+1) - 1\}\\ &= \{s(k+1) - i \mid i = 1, ..., s(k+1) - 1\} \setminus \{s(k+1) - i \mid i = k+1, 2(k+1), ..., (s-1)(k+1)\}. \end{aligned}$$

If we view T_0 as an interval, it is apparent that the interval is incomplete. Note what is missing:

$$\{s(k+1) - i \mid i = k+1, 2(k+1), ..., (s-1)(k+1)\}.$$

Recall that we defined I to be the the set of edges between consectutively named vertices on the spine and H to be the set of edges between non-consecutively named vertices on the spine. Since both I and H are affected by the size of the spine, there are two cases of each.

Case 1: s is even.

$$I = \{v_{2k+1}v_{2k+2}, v_{4k+3}v_{4k+4}, ..., v_{(s-2)(k+1)-1}v_{(s-2)(k+1)}\}.$$
$$H = \{v_0v_{2k+1}, v_{2k+2}v_{4k+3}, ..., v_{(s-2)(k+1)}v_{s(k+1)-1}\}.$$

Case 2: s is odd.

$$I = \{v_{2k+1}v_{2k+2}, v_{4k+3}v_{4k+4}, ..., v_{(s-1)(k+1)-1}v_{(s-1)(k+1)}\}.$$
$$H = \{v_0v_{2k+1}, v_{2k+2}v_{4k+3}, ..., v_{(s-3)(k+1)}v_{(s-1)(k+1)-1}\}.$$

Let I_0 be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in I$ and let H_0 denote the set of all $|f(u) - f(v)|$ for all $uv \in H$. The two cases follow from the sets above.

Case 1: s is even.

$$\begin{aligned} I_0 &= \{|[s(k+1) - \tfrac{2k+2}{2}] - \tfrac{2k+2}{2}|, |[s(k+1) - \tfrac{4k+4}{2}] - \tfrac{4k+4}{2}|, ..., |[s(k+1) - \tfrac{(s-2)(k+1)}{2}] - \tfrac{(s-2)(k+1)}{2}|\} \\ &= \{|s(k+1) - 2(k+1)|, |s(k+1) - 4(k+1)|, ..., |2(k+1)|\} \\ &= \{s(k+1) - 2(k+1), s(k+1) - 4(k+1), ..., 2(k+1)\}. \end{aligned}$$

$$\begin{aligned} H_0 &= \{|0 - [s(k+1) - \tfrac{2k+2}{2}]|, |\tfrac{2k+2}{2} - [s(k+1) - \tfrac{4k+4}{2}]|, ..., |\tfrac{(s-2)(k+1)}{2} - [s(k+1) - \tfrac{s(k+1)}{2}]|\} \\ &= \{|(k+1) - s(k+1)|, |3(k+1) - s(k+1)|, ..., |-(k+1)|\} \\ &= \{s(k+1) - (k+1), s(k+1) - 3(k+1), ..., k+1\}. \end{aligned}$$

Case 2: s is odd.

$$\begin{aligned} I_0 &= \{|[s(k+1) - \tfrac{2k+2}{2}] - \tfrac{2k+2}{2}|, |[s(k+1) - \tfrac{4k+4}{2}] - \tfrac{4k+4}{2}|, ..., |[s(k+1) - \tfrac{(s-1)(k+1)}{2}] - \tfrac{(s-1)(k+1)}{2}|\} \\ &= \{|s(k+1) - 2(k+1)|, |s(k+1) - 4(k+1)|, ..., |(k+1)|\} \\ &= \{s(k+1) - 2(k+1), s(k+1) - 4(k+1), ..., (k+1)\}. \end{aligned}$$

$$\begin{aligned} H_0 &= \{|0 - [s(k+1) - \tfrac{2k+2}{2}]|, |\tfrac{2k+2}{2} - [s(k+1) - \tfrac{4k+4}{2}]|, ..., |\tfrac{(s-3)(k+1)}{2} - [s(k+1) - \tfrac{(s-1)(k+1)}{2}]|\} \\ &= \{|(k+1) - s(k+1)|, |3(k+1) - s(k+1)|, ..., |-2(k+1)|\} \\ &= \{s(k+1) - (k+1), s(k+1) - 3(k+1), ..., 2(k+1)\}. \end{aligned}$$

In both cases, we obtain:

$$\begin{aligned} I_0 \cup H_0 &= \{s(k+1) - (k+1), s(k+1) - 2(k+1), ..., 2(k+1), k+1\} \\ &= \{s(k+1) - i \mid i = k+1, 2(k+1), ..., (s-1)(k+1)\}. \end{aligned}$$

Recall that this is the set missing from T_0 when viewing T_0 as an incomplete interval. Thus, $T_0 \cup I_0 \cup H_0$, the entire set of edge labels, is the complete interval $\{s(k+1) - i \mid i = 1, ..., s(k+1) - 1\}$. Notice that there are exactly $s(k+1) - 1$ unique integers in this set. Since each of these labels appears once on an edge, and there are $s(k+1) - 1$ edges, each label must appear exactly once.

Therefore, since the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(G)$ are all distinct, G is graceful. □

2.2 α-labeling

In this section, we will show that the graceful labeling produced above is, in fact, an α-labeling. Recall that for a graph G, a function $f : V(G) \to \{0, \ldots, |E(G)|\}$ is an α-labeling of G if f is a graceful labeling of G with the additional property that there exists some integer λ such that for all $uv \in E(G)$, $f(u) \leq \lambda$ and $f(v) > \lambda$. Equivalently, we can say that a graceful labeling of a bipartite graph with bipartition (A, B) is an α-labeling if there exists an integer λ such that $f(a) \leq \lambda$ for every $a \in A$ and $f(b) > \lambda$ for every $b \in B$. We use the latter definition to prove the following theorem.

Theorem 2.2. *The graceful labeling presented in Section 2.1 is an α-labeling.*

Proof. Let G be a uniform k-distant tree with n vertices, and let sets A and B be defined as follows:

$$A = \{v_i \mid i \text{ is even}\}$$
$$B = \{v_i \mid i \text{ is odd}\}$$

Note that both A and B are sets of vertices, but we will also refer to A and B as sets of indices where the indices are taken from the subscripts on the elements of A and B. Clearly, A and B are disjoint. To confirm that G is bipartite with bipartition (A, B), we will show that A and B are independent sets. Consider a vertex v_i. We have two cases: v_i is either a tail vertex or a spine vertex. If v_i is in the tail, by our naming procedure, we know that $N(v_i)$, the neighbor set of v_i, is a subset of $\{v_{i-1}, v_{i+1}\}$. If v_i is in the spine, $N(v_i) \subseteq \{v_{i-1}, v_{i+1}, v_{i+(2k+1)}, v_{i-(2k+1)}\}$. Notice that in any case, for any neighbor v_j of v_i, j is of the opposite parity of i. Thus, A and B must be independent sets. So, (A, B) forms a valid bipartition of G.

Notice that f is strictly increasing as the subscripts of the elements of A increase. Borrowing from calculus, we know then that $\max\{f(v_i) \mid v_i \in A\}$ occurs when i is largest in A. If n is even, i is largest when $i = n - 2$, and when n is odd, i is largest at $n - 1$. So, $f(a) \leq f(v_{n-2}) = \frac{n-2}{2}$ for every $a \in A$ when n is even, and $f(a) \leq f(v_{n-1}) = \frac{n-1}{2}$ for all $a \in A$ when n is odd.

Similarly, it is evident that f is strictly decreasing as the subscripts of the elements of B increase. So, we can conclude that $\min\{f(v_i) \mid v_i \in B\}$ occurs when i is largest. For n even, the largest value of i for v_i in B is $n - 1$, and hence, $f(b) \geq n - \frac{n}{2}$ for every $b \in B$. For n odd, the largest value of i is $n - 2$, so $f(b) \geq n - \frac{n-1}{2}$ for every $b \in B$.

Thus, when n is even, $f(a) \leq \frac{n-2}{2}$ for every $a \in A$, and $f(b) > \frac{n-2}{2}$ for every $b \in B$. When n is odd, $f(a) \leq \frac{n-1}{2}$ for every $a \in A$, and $f(b) > \frac{n-1}{2}$ for every $b \in B$. Therefore, our labeling is an α-labeling. □

3 Appending a-stars

To extend the previous result, for a graph G and postive integer a define $app_a(G)$ to be the graph obtained from appending a leaves to each leaf in G. In the following sections, we show that, if G is a uniform t-distant tree, $app_a(G)$ admits an alpha labeling.

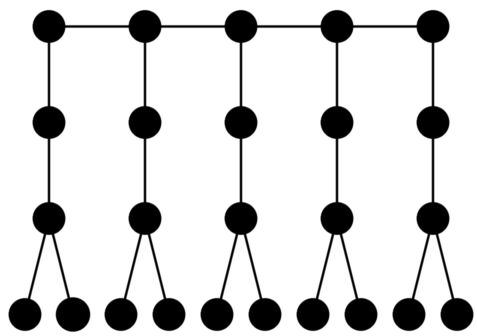

Figure 7: $app_a(G)$, where $a = 2$ and G is a uniform 2-distant tree.

3.1 Graceful Labeling

Let G be a uniform t-distant tree. To prove that $app_a(G)$ admits an α-labeling, we first establish that $app_a(G)$ is graceful. As when producing the function for uniform k-distant trees, we will name the vertices of $app_a(G)$ in a specific way before defining the function.

1. Arrange the vertices of $app_a(G)$ so that the tails extend downward from the spine. Notice that each leaf of G combined with the newly appended leaves produces an a star.

2. To begin, label the vertex in the upper left-hand corner v_0. Moving down the tail, label the next vertex v_1, the next v_2, etc., until the end of the first tail of G, i.e. v_t.

3. Continue naming the first leaf of the first tail with v_{t+1}. Proceed to name all leaves of the first tail, through v_{t+a}.

4. Continue naming the leaf directly to the right (on the next tail) with v_{t+a+1}, then the next leaf v_{t+a+2}, etc., until all leaves on this tail have been named, i.e. until a vertex has been named v_{t+2a}.

5. Proceed naming with v_{t+2a+1} on the adjacent tail vertex of G. Continue in this manner up the tail toward the spine.

6. After reaching the spine, move to the vertex directly to the right (on the spine), and continue down the next tail.

7. Proceed in this way until all vertices are named. The figure below exemplifies the procedure.

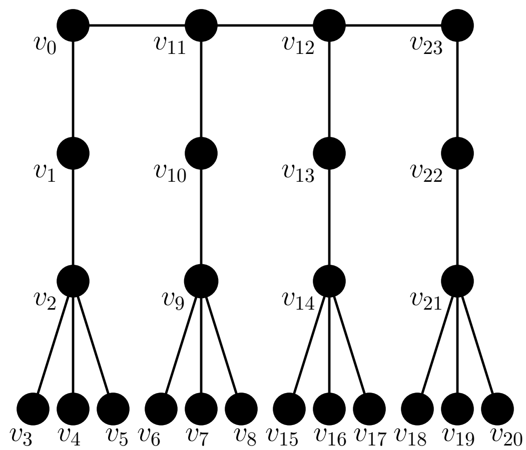

Figure 8: Naming the vertices of $app_3(G)$, where G is a uniform 2-distant tree.

To assign labels to the vertices, we define a function f from the n vertices of the graph to the set $\{0, 1, ..., n-1\}$, where $n-1$ is the number of edges. The function f is the same as that in Section 2.1 in all cases for vertices on the t-distant tree portion of the graph. If v_i is not a leaf, then

$$f(v_i) = \begin{cases} \dfrac{i}{2} & \text{if } i \text{ is even} \\ n - \dfrac{i+1}{2} & \text{if } i \text{ is odd} \end{cases}$$

However, to label the leaves, we split f into two cases. Notice that if v_i is a leaf, then i is contained in a closed interval of the form $[mt + (m-1)a + m, mt + ma + (m-1)]$ or $[mt + ma + m, mt + (m+1)a + (m-1)]$, where m is a positive, odd integer. We will refer to these intervals as Form 1 and Form 2, respectively. Let $j = mt + (m-1)a + m$, and let $l = mt + (m+1)a + (m-1)$. If v_i is a leaf, then:

Case 1: t is even.

$$f(v_i) = \begin{cases} n - (i - \frac{j-1}{2}) & \text{if } i \text{ is contained in an interval of Form 1} \\ i - \frac{l}{2} & \text{if } i \text{ is contained in an interval of Form 2} \end{cases}$$

Case 2: t is odd.

$$f(v_i) = \begin{cases} i - \frac{j}{2} & \text{if } i \text{ is contained in an interval of Form 1} \\ n - (i - \frac{l-1}{2}) & \text{if } i \text{ is contained in an interval of Form 2} \end{cases}$$

This function is best illustrated using an example.

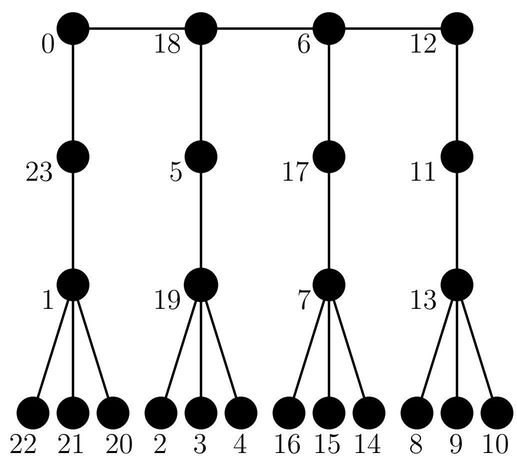

Figure 9: A graceful labeling of $app_3(G)$, where G is a uniform 2-distant tree.

We now prove that the function above is a graceful labeling for $app_a(G)$, where G is a t-distant tree.

Theorem 3.1. *Let a and t be positive integers. For every uniform t-distant tree G, $app_a(G)$ is graceful.*

Proof. Let a and t be positive integers and G be a uniform t distant tree. Let s be the number of vertices in the spine. Let $n = s(t+a+1)$. Notice that n is the number of vertices of $app_a(G)$.

Let $f : V(app_a(G)) \to \{0, 1, ..., n-1\}$, where $n-1$ is the number of edges of $app_a(G)$, be defined as in the function above. As in Section 2.1, we need to show that the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(app_a(G))$ are all distinct. We will use an approach similar to that of the proof of Theorem 2.1.

$E(app_a(G))$ can be viewed as the union of three sets of edges: edges in the tails of the t-distant tree, edges of the a-stars, and edges on the spine. We will let T denote the set of edges in the tails, I denote the set of edges in the stars, and H denote the set of edges in the spine. So, $E(app_a(G)) = T \cup I \cup H$.

Once again, we will determine what edge labels appear in $app_a(G)$ by looking at the edge labels appearing in each set T, I, and H individually. It is beneficial to examine these sets in two cases. We begin with the case that s is even. In this case,

$$\begin{aligned}T = &\{v_0v_1, v_1v_2, ..., v_{t-1}v_t\} \cup \{v_{t+2a+1}v_{t+2a+2}, v_{t+2a+2}v_{t+2a+3}, ..., v_{2t+2a}v_{2t+2a+1}\} \\ &\cup \{v_{2t+2a+2}v_{2t+2a+3}, v_{2t+2a+3}v_{2t+2a+4}, ..., v_{3t+2a+1}v_{3t+2a+2}\} \\ &\cup \{v_{3t+4a+3}v_{3t+4a+4}, v_{3t+4a+4}v_{3t+4a+5}, ..., v_{4t+4a+2}v_{4t+4a+3}\} \cup ... \\ &\cup \{v_{(s-1)t+sa+(s-1)}v_{(s-1)t+sa+s}, v_{(s-1)t+sa+s}v_{(s-1)t+sa+(s+1)}, ..., v_{s(t+a+1)-2}v_{s(t+a+1)-1}\}.\end{aligned}$$

Let T_0 be the set of induced edge labels given by $|f(u) - f(v)|$ for all $uv \in T$. When s is even, there are two cases of T_0.

Case 1: t is even.

$$\begin{aligned}T_0 = &\{|0 - [n-1]|, ..., |[n - \tfrac{t}{2}] - \tfrac{t}{2}|\} \cup \{|[n - \tfrac{t+2a+2}{2}] - \tfrac{t+2a+2}{2}|, \\ &..., |\tfrac{2t+2a}{2} - [n - \tfrac{2t+2a+2}{2}]|\} \cup \{|\tfrac{2t+2a+2}{2} - [n - \tfrac{2t+2a+4}{2}]|, ..., \\ &|[n - \tfrac{3t+2a+2}{2}] - \tfrac{3t+2a+2}{2}|\} \cup ... \cup \{|[n - \tfrac{(s-1)t+sa+s}{2}] - \tfrac{(s-1)t+sa+s}{2}|, \\ &..., |\tfrac{s(t+a+1)-2}{2} - [n - \tfrac{s(t+a+1)}{2}|\} \\ = &\{|1-n|, ..., |n-t|\} \cup \{|n - (t+2a+2)|, ..., \\ &|(2t+2a+1) - n|\} \cup \{|(2t+2a+3) - n|, ..., |n - (3t+2a+2)|\} \cup ... \cup \{|t|, ..., |-1|\}.\end{aligned}$$

Case 2: t is odd.

$$\begin{aligned}T_0 = &\{|0 - [n-1]|, ..., |\tfrac{t-1}{2} - [n - \tfrac{t+1}{2}]|\} \cup \{|\tfrac{t+2a+1}{2} - [n - \tfrac{t+2a+3}{2}|, \\ &..., |\tfrac{2t+2a}{2} - [n - \tfrac{2t+2a+2}{2}]|\} \cup \{|\tfrac{2t+2a+2}{2} - [n - \tfrac{2t+2a+4}{2}|, ..., \\ &|\tfrac{3t+2a+1}{2} - [n - \tfrac{3t+2a+3}{2}|\} \cup ... \cup \{|[n - \tfrac{(s-1)t+sa+s}{2}] - \tfrac{(s-1)t+sa+s}{2}|, \\ &..., |\tfrac{s(t+a+1)-2}{2} - [n - \tfrac{s(t+a+1)}{2}|\} \\ = &\{|1-n|, ..., |t-n|\} \cup \{|(t+2a+2) - n|, ..., \\ &|(2t+2a+1) - n|\} \cup \{|(2t+2a+3) - n|, ..., |(3t+2a+2) - n|\} \cup ... \cup \{|t|, ..., |-1|\}.\end{aligned}$$

Notice that upon further simplification, both of these cases of T_0 result in the same set. So, for any s even,

$$\begin{aligned}T_0 = &\{n-1, ..., n-t\} \cup \{n - (t+2a+2), ..., n - (2t+2a+1)\} \\ &\cup \{n - (2t+2a+3), ..., n - (3t+2a+2)\} \cup ... \cup \{t, ..., 1\}.\end{aligned}$$

Recall that we defined I to be the the set of edges in the stars. When s is even,

$$\begin{aligned}I = &\{v_tv_{t+1}, v_tv_{t+2}, ..., v_tv_{t+a}\} \cup \{v_{t+2a+1}v_{t+a+1}, v_{t+2a+1}v_{t+a+2}, ..., v_{t+2a+1}v_{t+2a}\} \\ &\cup \{v_{3t+2a+2}v_{3t+2a+3}, v_{3t+2a+2}v_{3t+2a+4}, ..., v_{3t+2a+2}v_{3t+3a+2}\} \cup ... \\ &\cup \{v_{(s-1)t+sa+(s-1)}v_{(s-1)t+(s-1)a+(s-1)}, v_{(s-1)t+sa+(s-1)}v_{(s-1)t+(s-1)a+s}, ..., \\ &\quad v_{(s-1)t+sa+(s-1)}v_{(s-1)t+sa+(s-2)}\}.\end{aligned}$$

Let I_0 be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in I$. As with T_0, there are initially two cases of I_0.

Case 1: t is even.

$$\begin{aligned}I_0 = &\{|\tfrac{t}{2} - [n-(t+1-\tfrac{t}{2})]|, ..., |\tfrac{t}{2} - [n-(t+a-\tfrac{t}{2})]|\} \cup \{|[n-\tfrac{t+2a+2}{2}] - [(t+a+1)-\tfrac{t+2a}{2}]|, ...,\\ &|[n-\tfrac{t+2a+2}{2}] - [(t+2a)-\tfrac{t+2a}{2}]|\} \cup \{|\tfrac{3t+2a+2}{2} - [n-(3t+2a+3-\tfrac{3t+2a+2}{2})]|, ...,\\ &|\tfrac{3t+2a+2}{2} - [n-(3t+3a+2-\tfrac{3t+2a+2}{2})]|\} \cup ... \cup\\ &\{|[n-\tfrac{(s-1)t+sa+s}{2}] - [(s-1)t+(s-1)a+(s-1)-\tfrac{(s-1)t+sa+(s-2)}{2}]|, ...,\\ &|[n-\tfrac{(s-1)t+sa+s}{2}] - [(s-1)t+sa+(s-2)-\tfrac{(s-1)t+sa+(s-2)}{2}]|\}\\ = &\{|(t+1)-n|, ..., |(t+a)-n|\} \cup \{|n-(t+a+2)|, ..., |n-(t+2a+1)|\} \cup \{|(3t+2a+3)-n|, ...,\\ &|(3t+3a+2)-n|\} \cup ... \cup \{|t+a|, ..., |t+1|\}.\end{aligned}$$

Case 2: t is odd.

$$\begin{aligned}I_0 = &\{|[n-\tfrac{t+1}{2}] - (t+1-\tfrac{t+1}{2})|, ..., |[n-\tfrac{t+1}{2}] - (t+a-\tfrac{t+1}{2})|\} \cup\\ &\{|\tfrac{t+2a+1}{2} - [n-(t+a+1-\tfrac{t+2a-1}{2})]|, ..., |\tfrac{t+2a+1}{2} - [n-(t+2a-\tfrac{t+2a-1}{2})]|\} \cup\\ &\{|[n-\tfrac{3t+2a+3}{2}] - [3t+2a+3-\tfrac{3t+2a+3}{2}]|, ..., |[n-\tfrac{3t+2a+3}{2}] - [3t+3a+2-\tfrac{3t+2a+3}{2}]|\\ &\cup ... \cup \{|\tfrac{(s-1)t+sa+(s-1)}{2} - [n-((s-1)t+(s-1)a+(s-1)-\tfrac{(s-1)t+sa+(s-2)-1}{2})]|,\\ &..., |\tfrac{(s-1)t+sa+(s-1)}{2} - [n-((s-1)t+sa+(s-2)-\tfrac{(s-1)t+sa+(s-2)-1}{2})]|\}\\ = &\{|n-(t+1)|, ..., |n-(t+a)|\} \cup \{|(t+a+2)-n|, ..., |(t+2a+1)-n|\} \cup \{|n-(3t+2a+3)|,\\ &..., |n-(3t+3a+2)|\} \cup ... \cup \{|-(t+a)|, ..., |-(t+1)|\}.\end{aligned}$$

In both cases, we obtain:

$$\begin{aligned}I_0 = &\{n-(t+1), ..., n-(t+a)\} \cup \{n-(t+a+2), ..., n-(t+2a+1)\}\\ &\cup \{n-(3t+2a+3), ..., n-(3t+3a+2)\} \cup ... \cup \{t+a, ...t+1\}.\end{aligned}$$

So, when s is even, we have that

$$\begin{aligned}T_0 \cup I_0 = &\{n-1, ..., n-(t+a)\} \cup \{n-(t+a+2), ..., n-(2t+2a+1)\}\\ &\cup \{n-(2t+2a+3), ..., n-(3t+3a+2)\} \cup ... \cup \{t+a, ..., 1\}\\ = &\{n-i \mid i = 1, ..., t+a\} \cup \{n-i \mid i = t+a+2, ..., 2t+2a+1\}\\ &\cup \{n-i \mid i = 2t+2a+3, ..., 3t+3a+2\} \cup ...\\ &\cup \{n-i \mid i = (s-1)t+(s-1)a+s, ..., s(t+a+1)-1\}\\ = &\{n-i \mid i = 1, ..., n-1\} \setminus \{n-i \mid i = t+a+1, 2(t+a+1), ..., (s-1)(t+a+1)\}\end{aligned}$$

When s is odd, this union takes shape a little differently. When s is odd,

$$\begin{aligned}T = &\{v_0v_1, v_1v_2, ..., v_{t-1}v_t\} \cup \{v_{t+2a+1}v_{t+2a+2}, v_{t+2a+2}v_{t+2a+3}, ..., v_{2t+2a}v_{2t+2a+1}\}\\ &\cup \{v_{2t+2a+2}v_{2t+2a+3}, v_{2t+2a+3}v_{2t+2a+4}, ..., v_{3t+2a+1}v_{3t+2a+2}\}\\ &\cup \{v_{3t+4a+3}v_{3t+4a+4}, v_{3t+4a+4}v_{3t+4a+5}, ..., v_{4t+4a+2}v_{4t+4a+3}\} \cup ...\\ &\cup \{v_{(s-1)t+(s-1)a+(s-1)}v_{(s-1)t+(s-1)a+s}, v_{(s-1)t+(s-1)a+s}v_{(s-1)t+(s-1)a+(s+1)}, ...,\\ &\quad v_{st+(s-1)a+(s-2)}v_{st+(s-1)a+(s-1)}\}.\end{aligned}$$

As when s is even, we have two cases of T_0 when s is odd.

Case 1: t is even.

$$\begin{aligned} T_0 = &\{|0-[n-1]|, ..., |[n-\tfrac{t}{2}]-\tfrac{t}{2}|\} \cup \{|[n-\tfrac{t+2a+2}{2}]-\tfrac{t+2a+2}{2}|, \\ &..., |\tfrac{2t+2a}{2}-[n-\tfrac{2t+2a+2}{2}]|\} \cup \{|\tfrac{2t+2a+2}{2}-[n-\tfrac{2t+2a+4}{2}]|, ..., \\ &|[n-\tfrac{3t+2a+2}{2}]-\tfrac{3t+2a+2}{2}|\} \cup ... \cup \{|\tfrac{(s-1)t+(s-1)a+(s-1)}{2}-[n-\tfrac{(s-1)t+(s-1)a+(s+1)}{2}]|, \\ &..., |[n-\tfrac{st+(s-1)a+(s-1)}{2}]-\tfrac{st+(s-1)a+(s-1)}{2}|\} \\ = &\{|1-n|, ..., |n-t|\} \cup \{|n-(t+2a+2)|, ..., |(2t+2a+1)-n|\} \cup \{|(2t+2a+3)-n|, \\ &..., |n-(3t+2a+2)|\} \cup ... \cup \{|-(t+a)|, ..., |a+1|\}. \end{aligned}$$

Case 2: t is odd.

$$\begin{aligned} T_0 = &\{|0-[n-1]|, ..., |\tfrac{t-1}{2}-[n-\tfrac{t+1}{2}]|\} \cup \{|\tfrac{t+2a+1}{2}-[n-\tfrac{t+2a+3}{2}|, \\ &..., |\tfrac{2t+2a}{2}-[n-\tfrac{2t+2a+2}{2}]|\} \cup \{|\tfrac{2t+2a+2}{2}-[n-\tfrac{2t+2a+4}{2}|, ..., \\ &|\tfrac{3t+2a+1}{2}-[n-\tfrac{3t+2a+3}{2}|\} \cup ... \cup \{\tfrac{(s-1)t+(s-1)a+(s-1)}{2}-[n-\tfrac{(s-1)t+(s-1)a+(s+1)}{2}]|, \\ &..., |\tfrac{(st+(s-1)a+(s-2)}{2}-[n-\tfrac{st+(s-1)a+s}{2}\} \\ = &\{|1-n|, ..., |t-n|\} \cup \{|(t+2a+2)-n|, ..., |(2t+2a+1)-n|\} \cup \{|(2t+2a+3)-n|, \\ &..., |(3t+2a+2)-n|\} \cup ... \cup \{|t+a|, ..., |-(a+1)|\}. \end{aligned}$$

Again, these reduce to the same set:

$$\begin{aligned} T_0 = &\{n-1, ..., n-t\} \cup \{n-(t+2a+2), ..., n-(2t+2a+1)\} \\ &\cup \{n-(2t+2a+3), ..., n-(3t+2a+2)\} \cup ... \cup \{t+a, ..., a+1\}. \end{aligned}$$

When s is odd,

$$\begin{aligned} I = &\{v_tv_{t+1}, v_tv_{t+2}, ..., v_tv_{t+a}\} \cup \{v_{t+2a+1}v_{t+a+1}, v_{t+2a+1}v_{t+a+2}, ..., v_{t+2a+1}v_{t+2a}\} \\ &\cup \{v_{3t+2a+2}v_{3t+2a+3}, v_{3t+2a+2}v_{3t+2a+4}, ..., v_{3t+2a+2}v_{3t+3a+2}\} \cup ... \\ &\cup \{v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+s}, v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+(s+1)}, ..., v_{st+(s-1)a+(s-1)}v_{s(t+a+1)-1}\}. \end{aligned}$$

Not surprisingly, we again have two cases of I_0.

Case 1: t is even.

$$\begin{aligned} I_0 = &\{|\tfrac{t}{2}-[n-(t+1-\tfrac{t}{2})]|, ..., |\tfrac{t}{2}-[n-(t+a-\tfrac{t}{2})]|\} \cup \{|[n-\tfrac{t+2a+2}{2}]-[(t+a+1)-\tfrac{t+2a}{2})]|, ..., \\ &|[n-\tfrac{t+2a+2}{2}]-[(t+2a)-\tfrac{t+2a}{2}]|\} \cup \{|\tfrac{3t+2a+2}{2}-[n-(3t+2a+3-\tfrac{3t+2a+2}{2})]|, ..., \\ &|\tfrac{3t+2a+2}{2}-[n-(3t+3a+2-\tfrac{3t+2a+2}{2})]|\} \cup ... \cup \\ &\{|\tfrac{st+(s-1)a+(s-1)}{2}-[n-(st+(s-1)a+s-\tfrac{st+(s-1)a+s-1}{2})]|, ..., \\ &|\tfrac{st+(s-1)a+(s-1)}{2}-[n-(s(t+a+1)-1-\tfrac{st+(s-1)a+s-1}{2})]|\} \\ = &\{|(t+1)-n|, ..., |(t+a)-n|\} \cup \{|n-(t+a+2)|, ..., |n-(t+2a+1)|\} \cup \{|(3t+2a+3)-n|, ..., \\ &|(3t+3a+2)-n|\} \cup ... \cup \{|a|, ..., |1|\}. \end{aligned}$$

Case 2: t is odd.

$$\begin{aligned}I_0 = \{&|[n - \tfrac{t+1}{2}] - (t+1-\tfrac{t+1}{2})|, ..., |[n-\tfrac{t+1}{2}] - (t+a-\tfrac{t+1}{2})|\} \cup \\ &\{|\tfrac{t+2a+1}{2} - [n-(t+a+1-\tfrac{t+2a-1}{2})]|, ..., |\tfrac{t+2a+1}{2} - [n-(t+2a-\tfrac{t+2a-1}{2})]|\} \cup \\ &\{|[n-\tfrac{3t+2a+3}{2}] - [3t+2a+3-\tfrac{3t+2a+3}{2}]|, ..., |[n-\tfrac{3t+2a+3}{2}] - [3t+3a+2-\tfrac{3t+2a+3}{2}]| \\ &\cup ... \cup \{|[n-\tfrac{st+(s-1)a+s}{2}] - [st+(s-1)a+s-\tfrac{st+(s-1)a+s}{2}]|, ..., \\ &|[n-\tfrac{st+(s-1)a+s}{2}] - [s(t+a+1)-1-\tfrac{st+(s-1)a+s}{2}]|\} \\ = \{&|n-(t+1)|, ..., |n-(t+a)|\} \cup \{|(t+a+2)-n|, ..., |(t+2a+1)-n|\} \cup \{|n-(3t+2a+3)|, \\ &..., |n-(3t+3a+2)|\} \cup ... \cup \{|a|, ..., |1|\}.\end{aligned}$$

So, when s is odd,

$$\begin{aligned}I_0 = &\{n-(t+1), ..., n-(t+a)\} \cup \{n-(t+a+2), ..., n-(t+2a+1)\} \\ &\cup \{n-(3t+2a+3), ..., n-(3t+3a+2)\} \cup ... \cup \{a, ...1\}.\end{aligned}$$

Notice that the union of T_0 and I_0 is the same as when s is even, i.e.

$$\begin{aligned}T_0 \cup I_0 = &\{n-1, ..., n-(t+a)\} \cup \{n-(t+a+2), ..., n-(2t+2a+1)\} \\ &\cup \{n-(2t+2a+3), ..., n-(3t+3a+2)\} \cup ... \cup \{t+a, ..., 1\} \\ = &\{n-i \mid i = 1, ..., t+a\} \cup \{n-i \mid i = t+a+2, ..., 2t+2a+1\} \\ &\cup \{n-i \mid i = 2t+2a+3, ..., 3t+3a+2\} \cup ... \\ &\cup \{n-i \mid i = (s-1)t+(s-1)a+s, ..., s(t+a+1)-1\} \\ = &\{n-i \mid i = 1, ..., n-1\} \setminus \{n-i \mid i = t+a+1, 2(t+a+1), ..., (s-1)(t+a+1)\}.\end{aligned}$$

If we view this union as an interval, note that we are missing the set $\{n-i \mid i = t+a+1, 2(t+a+1), ..., (s-1)(t+a+1)\}$. Recall that we named the set of edges in the spine H. Although they are similar, we will split H into two cases.

Case 1: s is even.

$$H = \{v_0v_{2t+2a+1}, v_{2t+2a+1}v_{2t+2a+2}, ..., v_{(s-2)(t+a+1)}v_{s(t+a+1)-1}\}.$$

Case 2: s is odd.

$$H = \{v_0v_{2t+2a+1}, v_{2t+2a+1}v_{2t+2a+2}, ..., v_{(s-1)(t+a+1)-1}v_{(s-1)(t+a+1)}\}.$$

Letting H_0 be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in H$, we have:

Case 1: s is even.

$$\begin{aligned}H_0 &= \{|0 - [n-\tfrac{2t+2a+2}{2}]|, |[n-\tfrac{2t+2a+2}{2}] - \tfrac{2t+2a+2}{2}|, ..., |\tfrac{(s-2)(t+a+1)}{2} - [n-\tfrac{s(t+a+1)}{2}]|\} \\ &= \{|(t+a+1)-n|, |n-2(t+a+1)|, ..., |-(t+a+1)|\}.\end{aligned}$$

Case 2: s is odd.

$$\begin{aligned}H_0 &= \{|0 - [n-\tfrac{2t+2a+2}{2}]|, |[n-\tfrac{2t+2a+2}{2}] - \tfrac{2t+2a+2}{2}|, ..., |[n-\tfrac{(s-1)(t+a+1)}{2}] - \tfrac{(s-1)(t+a+1)}{2}|\} \\ &= \{|(t+a+1)-n|, |n-2(t+a+1)|, ..., |t+a+1|\}.\end{aligned}$$

Again, these reduce to the same set:

$$\begin{aligned} H_0 &= \{n-(t+a+1), n-2(t+a+1), ..., t+a+1\} \\ &= \{n-i \mid i = t+a+1, 2(t+a+1), ..., (s-1)(t+a+1)\}. \end{aligned}$$

Recall that this is exactly the set we are missing from $T_0 \cup I_0$ when viewing it as an incomplete interval. Thus, $T_0 \cup I_0 \cup H_0$, the entire set of edge labels, is the complete interval $\{n-i \mid i = 1, ..., s(t+a+1)-1\}$. So, there are exactly $s(t+a+1)-1 = n-1$ unique integers in this interval, and each of these labels appears once on an edge. Since there are $n-1$ edges, we know that each edge label must appear exactly once. Therefore, since the induced edge labels given by $|f(u)-f(v)|$ for every $uv \in E(app_a(G))$ are all distinct, $app_a(G)$ is graceful. □

3.2 α-labeling

This labeling, like the labeling from Section 2, is also an α-labeling. We show this in a similar manner.

Theorem 3.2. *The graceful labeling presented in Section 3.1 is an α-labeling.*

Proof. Let a and t be positive integers and G be a uniform t-distant tree. By saying a vertex v is in G, we mean $v \in V(G)$. Note that any vertex in $app_a(G)$ is either in G, or it is a leaf. We will present this proof in two cases:

Case 1: t is even
Define sets A and B as follows:
$A = \{v_i \mid v_i \in G$ and i is even$\} \cup \{v_i \mid v_i$ is a leaf and i is contained in an interval of Form 2$\}$
$B = \{v_i \mid v_i \in G$ and i is odd$\} \cup \{v_i \mid v_i$ is a leaf and i is contained in an interval of Form 1$\}$

Note that both A and B are sets of vertices, but we will also refer to A and B as sets of indices where the indices are taken from the subscripts on the elements of A and B. We can see that A and B are disjoint. To be sure $app_a(G)$ is bipartite with bipartition (A, B), we must show that A and B are independent. Consider a vertex $v_i \in A$. This vertex must satisfy one of three cases: v_i could be in G but not adjacent to a leaf, v_i could be in G and adjacent to a leaf, or it could be a leaf. If v_i is in G but not adjacent to a leaf, then by our naming scheme, $N(v_i)$, the neighbor set of v_i, is a subset of $\{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$. In this case, any neighbor v_j of v_i would be in B, since j must be in G and is of opposite parity than i.

If v_i is in G and is adjacent to a leaf, then $N(v_i) = \{v_{i-1}\} \cup \{v_{i+1}, ..., v_{i+a}\}$, where $v_{i-1} \in G$. Note that according to our naming procedure, i is of the form $mt + (m-1)a + (m-1)$ for some odd integer m. Thus, $N(v_i) = \{v_{i-1}\} \cup \{v_{mt+(m-1)a+m}, ..., v_{mt+ma+(m-1)}\}$. Thus, for any neighbor v_j of v_i, either j is in G and is of opposite parity than i, or j is in an interval of Form 1. Thus, v_j must be in B.

Finally, consider v_i, where v_i is a leaf. Since $v_i \in A$, i must be contained in an interval of Form 2, i.e. $i \in \{mt + ma + m, ..., mt + (m+1)a + (m-1)\}$ for some positive, odd integer m. Note also that for each unique m, $N(v_i) = \{v_{mt+(m+1)a+m}\}$ for every $i \in \{mt + ma +$

$m, ..., mt + (m + 1)a + (m - 1)\}$. Since t is even, we know that $mt + (m + 1)a + m$ must be odd. Further, $v_{mt+(m+1)a+m} \in G$. Therefore, $N(v_i) \subseteq B$ for every $v_i \in A$ where v_i is a leaf.

Similarly, consider $v_i \in B$. If v_i is in G but not adjacent to a leaf, again $N(v_i) \subseteq \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$, so any neighbor v_j of v_i must be in A. If v_i is in G and is adjacent to a leaf, $N(v_i) = \{v_{i+1}\} \cup \{v_{i-1}, ..., v_{i-a}\}$, where $v_{i+1} \in G$. In this case, when v_i is adjacent to a leaf, i must be of the form $mt + (m + 1)a + m$, revealing that any neighbor of v_i is in A. If v_i is a leaf, then $i \in \{mt + (m - 1)a + m, ..., mt + ma + (m - 1)\}$ for some positive, odd integer m, and $N(v_i) = \{v_{mt+(m-1)a+(m-1)}\}$. Thus, it is clear that $N(v_i) \subseteq A$ when $v_i \in B$. Therefore, A and B must be independent sets of vertices.

Now, notice that $f(x)$ increases as i increases in A. Thus, $\max\{f(v_i) \mid v_i \in A\}$ must occur where i is largest. When s is even, the largest value of i in A is $n-2$, where $v_{n-2} \in G$. So, for every $x \in A$, $f(x) \leq \frac{n-2}{2}$. When s is odd, i is largest in A where $i = n - (a + 1)$, where v_i is in G. Thus, $f(x) \leq \frac{n-(a+1)}{2}$ for all $x \in A$.

On the other hand, $f(b)$ is decreasing as the subscripts in B increase; hence, $\min\{f(v_i) \mid v_i \in B\}$ occurs where i is largest in B. When s is even, the largest subscript found in B is $n - 1$, where $v_{n-1} \in G$. Thus, for all $b \in B, f(b) \geq n - \frac{n}{2} = \frac{n}{2} > \frac{n-2}{2}$. When s is odd, i is largest in B where $i = n - 1$, where v_i is a leaf. Therefore, when s is odd, $f(b) \geq f(v_{n-1}) = \frac{n-(a-1)}{2} > \frac{n-(a+1)}{2}$ for every $b \in B$.

Case 2: t is odd
$A = \{v_i \mid v_i \in G \text{ and } i \text{ is even}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 1}\}$
$B = \{v_i \mid v_i \in G \text{ and } i \text{ is odd}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 2}\}$

Again, note that both A and B are sets of vertices, but we will also refer to A and B as sets of indices where the indices are taken from the subscripts on the elements of A and B. While it is clear that A and B are disjoint, we must show that they are independent to confirm that $app_a(G)$ is bipartite with bipartition (A, B). Consider a vertex $v_i \in A$. If v_i is in G but not adjacent to a leaf, we know by our naming scheme that $N(v_i) \subseteq \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$. So, any vertex adjacent to v_i would have a subscript of opposite parity as i and would thus be in B. If v_i is in G but adjacent to a leaf, $N(v_i) = \{v_{i+1}\} \cup \{v_{i-1}, ...v_{i-a}\}$. According to our naming procedure, i must be of the form $mt + (m + 1)a + m$ for some positive, odd integer m. Thus, $N(v_i) = \{v_{i+1}\} \cup \{v_{mt+(m+1)a+(m-1)}, ..., v_{mt+ma+m}\}$, so any neighbor of v_i must be in B. If v_i is a leaf, then $i \in \{mt + (m - 1)a + m, ..., mt + ma + (m - 1)\}$ for a positive, odd integer m, and $N(v_i) = \{v_{mt+(m-1)a+(m-1)}\}$. Thus, it is clear that $N(v_i) \subseteq B$ when $v_i \in A$.

Now let $v_i \in B$. If $v_i \in G$ and is not adjacent to a leaf, again we have that $N(v_i) \subseteq \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$, so any neighbor of v_i would be in A. If v_i is in G and adjacent to a leaf, then $N(v_i) = \{v_{i-1}\} \cup \{v_{i+1}, ..., v_{i+a}\}$, where $v_{i-1} \in G$. Since, by our naming procedure, i is of the form $mt+(m-1)a+(m-1)$, it's easy to see that any neighbor of v_i would be in A. If v_i is a leaf, then $i \in \{mt + ma + m, ..., mt + (m + 1)a + (m - 1)\}$ for some positive, odd integer m, and $N(v_i) = v_{mt+(m+1)a+m} \in G$. Thus, since $mt+(m+1)a+m$ is even, we know that $N(v_i) \in A$. Therefore, A and B are independent sets of vertices.

Notice that f is increasing as the subscripts of the elements of A increase. Again, we can conclude that $\max\{f(v_i) \mid v_i \in A\}$ occurs when i is largest. When s is even, i is largest when

$i = n-2$, where $v_i \in G$. When s is odd, i is largest at $n-1$, where v_i is a leaf. So, $f(x) \leq \frac{n-2}{2}$ for every $x \in A$ when s is even and $f(x) \leq s(t+a+1) - (\frac{st+1+(s-1)a+(s-1)}{2}) = \frac{n+(a-2)}{2}$ for every $x \in A$ when s is odd.

Similarly, f is decreasing as subscripts of the elements of B decrease. Thus, we know that $\min\{f(v_i) \mid v_i \in B\}$ occurs when i is largest. For s even, the largest value of i for v_i in B is $n-1$, where v_i is in G, so $f(b) \geq n - \frac{n}{2} > \frac{n-2}{2}$ for every $b \in B$. When s is odd, the largest valued subscript in B is $i = n-(a+1)$, where v_i is in G. Thus, $f(b) \geq n - \frac{n-(a+1)+1}{2} = \frac{n+a}{2} > \frac{n+(a-2)}{2}$ for every $b \in B$.

We have shown that in every case, for bipartition (A, B) of $app_a(G)$, there exists an integer λ such that $f(x) \leq \lambda$ for every $x \in A$ and $f(b) > \lambda$ for every $b \in B$. Therefore, our labeling is an α-labeling.

□

4 Concluding Remarks

Our labeling method has been successful for obtaining harmonious, graceful, and α-labelings of uniform k-distant trees, as well as graceful and α-labelings of those same graphs with stars appended to the ends of the tails. We hope that the same method can be used to show that all uniform k-distant trees with stars appended are harmonious. It is noted by Gallian in [5] that whether or not lobsters are harmonious seems to have attracted no attention thus far. It would be interesting to see if a similar approach works for showing that all lobsters are harmonious. Another problem to consider is to allow for different numbers of leaves to be appended to each tail of a uniform k-distant tree, and to find an α-labeling for this class of graphs.

APPENDIX

The following definitions may be useful to reference. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

Two vertices are *adjacent* if they are connected by an edge.

A *path* in G is a sequence of distinct vertices such that there is an edge between each set of consecutive vertices.

A *cycle* is a path that begins and ends with the same vertex. If G has no cycles, then we call it an *acyclic* graph.

G is *connected* if and only if every pair of vertices in $V(G)$ can be joined by a path in G.

A *tree* is a connected, acyclic graph.

The *degree* of a vertex is the number of edges incident to that vertex. A vertex of degree one is called a *leaf*.

A set of vertices for which there is no edge with both endpoints in that set is called an *independent set of vertices.*

G is *bipartite* with bipartition (A, B) if $V(G)$ can be partitioned into two disjoint subsets A and B such that A and B are each independent sets of vertices.

Let H be a graph with vertex set $V(H)$ and edge set $E(H)$. Graphs G and H are *isomorphic* (denoted $G \cong H$) if there exists a bijection $f : V(G) \to V(H)$ such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$. The function f is called an *isomorphism*. An isomorphism of a graph to itself is called an *automorphism*.

C_n denotes the cycle on n vertices.

A *complete graph* is a graph in which every pair of distinct vertices is connected by an edge. K_n denotes the complete graph on n vertices.

$\mathbb{Z}_n$ denotes the group of integers modulo n.

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