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Two Rosa-type Labelings of Uniform k-distant Trees and a New Class of Trees

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Abstract

A $k$-distant tree consists of a main path, called the spine, such that each vertex on the spine is joined by an edge to an end-vertex of at most one path on at most $k$ vertices. Those paths, along with the edge joining them to the spine, are called tails. When every vertex on the spine has exactly one incident tail of length $k$ we call the tree a uniform $k$-distant tree. We show that every uniform $k$-distant tree admits both a graceful- and an $\alpha$-labeling.

For a graph $G$ and a positive integer $a$, define $app_a(G)$ to be the graph obtained from appending $a$ leaves to each leaf in $G$. When $G$ is a uniform $k$-distant tree, we show that $app_a(G)$ admits both a graceful- and an $\alpha$-labeling.

1 Introduction

Let $G$ be a graph. Denote the vertex set and edge set of $G$ by $V(G)$ and $E(G)$, respectively. A $k$-distant tree consists of a main path, called the spine, such that each vertex on the spine is joined by an edge to an end-vertex of at most one path on at most $k$ vertices. Those paths, along with the edge joining them to the spine, are called tails. When every vertex on the spine has exactly one incident tail of length $k$ we call the tree a uniform $k$-distant tree.  

A graceful labeling of a graph $G$ on $n$ vertices is a one-to-one function from the vertices of $G$ to the set $\{0, \ldots, |E(G)|\}$ such that the induced edge labels given by $|f(u) - f(v)|$, for every $uv \in E(G)$, are all distinct. If a graph admits a graceful labeling then that graph is said to be graceful. Graceful labelings were first introduced by Rosa in 1967 [10] to generate graph decompositions. As a historical note, Rosa used the term $\beta$-valuation for what is now commonly known as a graceful labeling. The term “graceful labeling” was coined years later.

Figure 1: An example of a uniform 3-distant tree.

\footnote{Fundamental definitions can be found in the appendix.}
Let $G$ and $H$ be graphs. A $G$-decomposition of $H$ is a set of edge-disjoint subgraphs of $H$, $\mathcal{D} = \{G_1, G_2, \ldots, G_t\}$ such that $\bigcup_{i=1}^{t} G_i = E(H)$ and for each $1 \leq i \leq t$, $G_i \cong G$. Identify the vertices of $K_n$ with the elements of $\mathbb{Z}_n$. Now, consider a $G$-decomposition of $K_n$, $\mathcal{D}$. If applying the permutation $(0, 1, 2, \ldots, n-1)$ to the vertices of the elements in $\mathcal{D}$ is an automorphism of $\mathcal{D}$ then we say that $\mathcal{D}$ is cyclic. In 1963 Ringel conjectured that for every tree $T$ on $n$ edges there exists a $T$-decomposition of $K_{2n+1}$, the complete graph on $2n + 1$ vertices. It was further conjectured by Kötzing that not only does $K_{2n+1}$ admit a $T$-decomposition, but it also admits a cyclic $T$-decomposition. The conjecture that for every tree, $T$, on $n$ edges there exists a cyclic $T$-decomposition of $K_{2n+1}$ is known as the Ringel-Kötzig conjecture. The connection between graph labelings and cyclic graph decompositions is as shown in the following theorem.

**Theorem 1.1** (Rosa [10]). Let $G$ be a graph with $n$ edges. If $G$ is graceful then a cyclic $G$-decomposition of $K_{2n+1}$ exists.

Combining Theorem 1.1 with the Ringel-Kötzig conjecture yields the following famous conjecture, which has motivated many of the results on graceful labelings.

**Conjecture 1.2** (Graceful Tree Conjecture). All trees are graceful.

For a given tree, Theorem 1.1 provides one graph decomposition. In order to strengthen this result, we must visit another labeling that was defined in Rosa’s original paper. For a graph $G$, we call a function $f : V(G) \rightarrow \{0, \ldots, |E(G)|\}$ an $\alpha$-labeling of $G$ if $f$ is a graceful labeling of $G$ with the additional property that there exists some integer $\lambda$ such that for all $uv \in E(G)$, $f(u) \leq \lambda$ and $f(v) > \lambda$. Notice that if $G$ admits an $\alpha$-labeling, then $G$ is necessarily bipartite. The connection between $\alpha$-labelings and cyclic graph decompositions is as shown below. Note that $\alpha$-labelings lead to an infinite number of graph decompositions.

**Theorem 1.3** (Rosa [10]). Let $G$ be a graph with $n$ edges. If $G$ admits an $\alpha$-labeling then for every positive integer $x$ there exists a cyclic $G$-decomposition of $K_{2nx+1}$.

While graph labelings continue to be used in the study of graph decompositions, they have since become a thoroughly studied subject of their own. A dynamic survey [5] is maintained by Gallian and contains over 1400 references. Thus far, not many general results have been obtained in order to attack the Graceful Tree Conjecture. It is common practice to show that various classes of trees are graceful. To this end, we define the following classes of trees. A caterpillar is a tree of order three or more in which the removal of its leaves produces a path.
Figure 3: A graceful labeling of $C_4$ (left), along with the corresponding embedding of $C_4$ into $K_9$ that generates a cyclic decomposition. Notice that, in this case, the graceful labeling is also an $\alpha$-labeling.

A lobster is a tree with the property that the removal of its leaves produces a caterpillar. We note that uniform 1-distance trees are special caterpillars and uniform 2-distance trees are special lobsters. It is known that all caterpillars are graceful [10], and that all lobsters with a perfect matching are graceful [7], yet it remains an open problem to determine if all lobsters are graceful. Murugan showed in [9] that, among admitting other labelings, all uniform $k$-distance trees admit a graceful labeling. Aldred and McKay [2] have shown that all trees on at most 27 vertices are graceful. Fang [3] has extended this to include all trees on at most 35 vertices.

In 1980, Graham and Sloane defined the following graph labeling as a means of obtaining additive bases of sets of integers. A harmonious labeling of a graph $G$ is a function $f : V(G) \rightarrow \{0, ..., |E(G)|-1\}$ such that $f$ is injective, and the induced edge labels given by $(f(u) + f(v)) \mod m$, for every $uv \in E(G)$, are all distinct. In the case where $G$ is a tree, exactly one vertex label can be repeated exactly once.

Figure 4: An example of a harmonious labeling of a graph.

An analogous version of the Graceful Tree Conjecture has been formulated for harmonious labelings.
Conjecture 1.4 (Graham and Sloan [6], 1980). All trees are harmonious.

Some results have been obtained to positively support this conjecture. Aldred and McKay [2] have shown that all trees on at most 26 vertices are harmonious, and this was extended to all trees on at most 31 vertices by Fang [4]. Graham and Sloane showed in [6] that all caterpillars are harmonious. It is not known if all lobsters are harmonious. In [1], it is shown that every uniform \(k\)-distant tree with an even number of vertices admits a harmonious labeling. These results were extended in [8] to include all uniform \(k\)-distant trees as harmonious graphs. Our current research is motivated by the previous two results on uniform \(k\)-distant trees, along with the Graceful Tree conjecture.

2 Uniform \(k\)-distant Trees

2.1 Graceful Labeling

In this section we produce a function which is a graceful labeling of a uniform \(k\)-distant tree. To define the function, it is necessary for us to name the vertices of the uniform \(k\)-distant tree in a specific way. Name the vertices as follows.

1. Arrange the vertices of the tree so that the tails extend downward from the spine.

2. To begin, label the vertex in the upper left-hand corner \(v_0\). Moving down the tail, label the next vertex \(v_1\), the next \(v_2\), etc., until the end of the tail is reached, i.e. \(v_k\).

3. Continue naming the vertex directly to the right (on the next tail) with \(v_{k+1}\). Proceed in the same way up the tail toward the spine.

4. After reaching the spine, or the vertex \(v_{2k+1}\), move to the vertex directly to the right (on the spine), and continue down the next tail.

5. Proceed in this way until all vertices are named. The figure below exemplifies the procedure.

![Figure 5: Naming the vertices of a uniform 3-distant tree.](image-url)
To assign labels to the vertices, we define a function $f$ from the $n$ vertices of the graph to the set $\{0, 1, ..., n-1\}$, where $n-1$ is the number of edges.

$$f(v_i) = \begin{cases} 
    \frac{i}{2} & \text{if } i \text{ is even} \\
    n - \frac{i+1}{2} & \text{if } i \text{ is odd}
\end{cases}$$

![Figure 6: A graceful labeling of the vertices of a uniform 3-distant tree.](image)

We will now show that given a uniform $k$-distant tree, the function described above is a graceful labeling of that tree.

**Theorem 2.1.** Every uniform $k$-distant tree is graceful.

**Proof.** Let $G$ be a uniform $k$-distant tree, where $k$ is the number of edges in each tail, and let $s$ denote the number of vertices in the spine. Note that $n = s(k+1)$, where $n$ is the number of vertices of $G$. Let $V(G)$ be the set of vertices of $G$ and $E(G)$ be the set of edges of $G$.

Let $f : V(G) \rightarrow \{0, 1, ..., n-1\}$, where $n-1$ is the number of edges of $G$, be defined as in the algorithm above. To prove that $G$ is graceful, we must show that the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(G)$ are all distinct.

We can view $E(G)$ as the union of three sets of edges: edges in the tails, edges between consecutively named vertices on the spine (i.e. $v_{2k+1}v_{2k+2}$), and edges between non-consecutively named vertices on the spine. Let $T$ denote the set of edges in the tails, $I$ denote the set of edges between consecutively named vertices on the spine, and $H$ denote the set of edges between non-consecutively named vertices on the spine. So, $E(G) = T \cup I \cup H$.

Let us examine each set making up $E(G)$ more closely. $T$ is the set of edges in the tails. Thus,

$$T = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\} \cup \{v_{k+1}v_{k+2}, v_{k+2}v_{k+3}, ..., v_{2k}v_{2k+1}\}$$

$$\cup \{v_{2k+2}v_{2k+3}, v_{2k+3}v_{2k+4}, ..., v_{3k+1}v_{3k+2}\} \cup \{v_{3k+3}v_{3k+4}, v_{3k+4}v_{3k+5}, ..., v_{4k+2}v_{4k+3}\}$$

$$\cup ... \cup \{v_{(s-1)(k+1)}v_{(s-1)(k+1)+1}, v_{(s-1)(k+1)+1}v_{(s-1)(k+1)+2}, ..., v_{s(k+1)-2}v_{s(k+1)-1}\}.$$
Let \( T_0 \) be the set of induced edge labels given by \( |f(u) - f(v)| \) for all \( uv \in T \). Here we examine the four cases of \( T_0 \).

**Case 1:** Both \( k \) and \( s \) are even.

\[
T_0 = \{ \lfloor 0 - [s(k+1) - 1] \rfloor, \ldots, \lfloor s(k+1) - \frac{k}{2} \rfloor - \frac{k}{2} \} \cup \{ \lfloor s(k+1) - \frac{k+2}{2} \rfloor - \frac{k+2}{2}, \ldots, \lfloor 2k - [s(k+1) - \frac{2k+2}{2}] \rfloor \cup \{ \lfloor 2k+2 - [s(k+1) - \frac{2k+4}{2}] \rfloor, \ldots, \lfloor s(k+1) - \frac{3k+2}{2} \rfloor - \frac{3k+2}{2} \} \cup \ldots \cup \{ \lfloor s(k+1) - \frac{(s-1)(k+1)+1}{2} \rfloor - \frac{(s-1)(k+1)+1}{2}, \ldots, \lfloor s(k+1) - \frac{s(k+1)+2}{2} \rfloor - \frac{s(k+1)+2}{2} \}\}
\]

\[
= \{ [1 - s(k+1)], \ldots, [s(k+1) - k] \} \cup \{ [s(k+1) - (k+2)], \ldots, |(2k+1) - s(k+1)| \} \cup \{ |(2k+3) - s(k+1)|, \ldots, |s(k+1) - (3k+2)| \} \cup \ldots \cup \{ |k|, \ldots, |1| \}.
\]

**Case 2:** \( k \) is even, \( s \) is odd.

\[
T_0 = \{ \lfloor 0 - [s(k+1) - 1] \rfloor, \ldots, \lfloor s(k+1) - \frac{k}{2} \rfloor - \frac{k}{2} \} \cup \{ \lfloor s(k+1) - \frac{k+2}{2} \rfloor - \frac{k+2}{2}, \ldots, \lfloor 2k - [s(k+1) - \frac{2k+2}{2}] \rfloor \cup \{ \lfloor 2k+2 - [s(k+1) - \frac{2k+4}{2}] \rfloor, \ldots, \lfloor s(k+1) - \frac{3k+2}{2} \rfloor - \frac{3k+2}{2} \} \cup \ldots \cup \{ \lfloor (s-1)(k+1) \rfloor - \frac{(s-1)(k+1)+2}{2}, \ldots, \lfloor s(k+1) - \frac{s(k+1)+1}{2} \rfloor - \frac{s(k+1)+1}{2} \}\}
\]

\[
= \{ [1 - s(k+1)], \ldots, [s(k+1) - k] \} \cup \{ [s(k+1) - (k+2)], \ldots, |(2k+1) - s(k+1)| \} \cup \{ |(2k+3) - s(k+1)|, \ldots, |s(k+1) - (3k+2)| \} \cup \ldots \cup \{ |-k|, \ldots, |1| \}.
\]

**Case 3:** Both \( k \) and \( s \) are odd.

\[
T_0 = \{ \lfloor 0 - [s(k+1) - 1] \rfloor, \ldots, \lfloor k+1 - [s(k+1) - \frac{k+1}{2}] \rfloor \} \cup \{ \lfloor k+1 - [s(k+1) - \frac{k+3}{2}] \rfloor, \ldots, \lfloor 2k - [s(k+1) - \frac{2k+2}{2}] \rfloor \cup \{ \lfloor 2k+2 - [s(k+1) - \frac{2k+4}{2}] \rfloor, \ldots, \lfloor 3k+1 - [s(k+1) - \frac{3k+3}{2}] \rfloor \cup \ldots \cup \{ \lfloor (s-1)(k+1) \rfloor - \frac{(s-1)(k+1)+2}{2}, \ldots, \lfloor s(k+1) - \frac{s(k+1)+2}{2} \rfloor - \frac{s(k+1)+2}{2} \}\}
\]

\[
= \{ [1 - s(k+1)], \ldots, [k - s(k+1)] \} \cup \{ [(k+2) - s(k+1)], \ldots, |(2k+1) - s(k+1)| \} \cup \{ |(2k+3) - s(k+1)|, \ldots, |3k+2 - s(k+1)| \} \cup \ldots \cup \{ |-k|, \ldots, |1| \}.
\]

**Case 4:** \( k \) is odd, \( s \) is even.

\[
T_0 = \{ \lfloor 0 - [s(k+1) - 1] \rfloor, \ldots, \lfloor k+1 - [s(k+1) - \frac{k+1}{2}] \rfloor \} \cup \{ \lfloor k+1 - [s(k+1) - \frac{k+3}{2}] \rfloor, \ldots, \lfloor 2k - [s(k+1) - \frac{2k+2}{2}] \rfloor \cup \{ \lfloor 2k+2 - [s(k+1) - \frac{2k+4}{2}] \rfloor, \ldots, \lfloor 3k+1 - [s(k+1) - \frac{3k+3}{2}] \rfloor \cup \ldots \cup \{ \lfloor (s-1)(k+1) \rfloor - \frac{(s-1)(k+1)+2}{2}, \ldots, \lfloor s(k+1) - \frac{s(k+1)+2}{2} \rfloor - \frac{s(k+1)+2}{2} \}\}
\]

\[
= \{ [1 - s(k+1)], \ldots, [k - s(k+1)] \} \cup \{ [(k+2) - s(k+1)], \ldots, |(2k+1) - s(k+1)| \} \cup \{ |(2k+3) - s(k+1)|, \ldots, |3k+2 - s(k+1)| \} \cup \ldots \cup \{ |-k|, \ldots, |1| \}.
\]

Notice that upon further simplification, each case of \( T_0 \) results in the same set. So, for any \( k \) and any \( s \),

\[
T_0 = \{ s(k+1) - 1, \ldots, s(k+1) - k \} \cup \{ s(k+1) - (k+2), \ldots, s(k+1) - (2k+1) \}
\]

\[
\cup \{ s(k+1) - (2k+3), \ldots, s(k+1) - (3k+2) \} \cup \ldots \cup \{ k, \ldots, 1 \}
\]

\[
= \{ s(k+1) - i \mid i = 1, \ldots, k \} \cup \{ s(k+1) - i \mid i = k+2, \ldots, 2k+1 \} \cup \{ s(k+1) - i \mid i = 2k+3, \ldots, 3k+2 \} \cup \ldots \cup \{ s(k+1) - i \mid i = s(k+1) - k, \ldots, s(k+1) - 1 \}
\]

\[
= \{ s(k+1) - i \mid i = 1, \ldots, s(k+1) - 1 \} \setminus \{ s(k+1) - i \mid i = k+1, 2(k+1), \ldots, (s-1)(k+1) \}.
\]
If we view $T_0$ as an interval, it is apparent that the interval is incomplete. Note what is missing:
\[
\{s(k + 1) - i \mid i = k + 1, 2(k + 1), \ldots, (s - 1)(k + 1)\}.
\]

Recall that we defined $I$ to be the set of edges between consecutively named vertices on the spine and $H$ to be the set of edges between non-consecutively named vertices on the spine. Since both $I$ and $H$ are affected by the size of the spine, there are two cases of each.

**Case 1:** $s$ is even.
\[
I = \{v_{2k+1}v_{2k+2}, v_{4k+3}v_{4k+4}, \ldots, v_{(s-2)(k+1)-1}v_{(s-2)(k+1)}\},
H = \{v_0v_{2k+1}, v_{2k+2}v_{4k+3}, \ldots, v_{(s-2)(k+1)}v_{(s-2)(k+1)-1}\}.
\]

**Case 2:** $s$ is odd.
\[
I = \{v_{2k+1}v_{2k+2}, v_{4k+3}v_{4k+4}, \ldots, v_{(s-1)(k+1)-1}v_{(s-1)(k+1)}\},
H = \{v_0v_{2k+1}, v_{2k+2}v_{4k+3}, \ldots, v_{(s-3)(k+1)}v_{(s-3)(k+1)-1}\}.
\]

Let $I_0$ be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in I$ and let $H_0$ denote the set of all $|f(u) - f(v)|$ for all $uv \in H$. The two cases follow from the sets above.

**Case 1:** $s$ is even.
\[
I_0 = \{|[s(k + 1) - \frac{2k+2}{2}] - \frac{2k+2}{2}, [s(k + 1) - \frac{4k+4}{2}] - \frac{4k+4}{2}, \ldots, [s(k + 1) - \frac{(s-2)(k+1)}{2}] - \frac{(s-2)(k+1)}{2}\|
= \{|s(k + 1) - 2(k + 1)|, |s(k + 1) - 4(k + 1)|, \ldots, |2(k + 1)|\}
= \{s(k + 1) - 2(k + 1), s(k + 1) - 4(k + 1), \ldots, 2(k + 1)\}.
\]

\[
H_0 = \{|0 - [s(k + 1) - \frac{2k+2}{2}], \frac{2k+2}{2} - [s(k + 1) - \frac{4k+4}{2}], \ldots, \frac{(s-2)(k+1)}{2} - s(k + 1)|\}
= \{|0 - [s(k + 1) - 2(k + 1)], \frac{2k+2}{2} - [s(k + 1) - 4(k + 1)], \ldots, |2(k + 1)|\}
= \{s(k + 1) - (k + 1), s(k + 1) - 3(k + 1), \ldots, k + 1\}.
\]

**Case 2:** $s$ is odd.
\[
I_0 = \{|[s(k + 1) - \frac{2k+2}{2}] - \frac{2k+2}{2}, [s(k + 1) - \frac{4k+4}{2}] - \frac{4k+4}{2}, \ldots, [s(k + 1) - \frac{(s-1)(k+1)}{2}] - \frac{(s-1)(k+1)}{2}\|
= \{|s(k + 1) - 2(k + 1)|, |s(k + 1) - 4(k + 1)|, \ldots, |k + 1|\}
= \{s(k + 1) - 2(k + 1), s(k + 1) - 4(k + 1), \ldots, (k + 1)\}.
\]

\[
H_0 = \{|0 - [s(k + 1) - \frac{2k+2}{2}], \frac{2k+2}{2} - [s(k + 1) - \frac{4k+4}{2}], \ldots, \frac{(s-3)(k+1)}{2} - s(k + 1)|\}
= \{|0 - [s(k + 1) - 2(k + 1)], \frac{2k+2}{2} - [s(k + 1) - 4(k + 1)], \ldots, |2(k + 1)|\}
= \{s(k + 1) - (k + 1), s(k + 1) - 3(k + 1), \ldots, 2(k + 1)\}.
\]

In both cases, we obtain:
\[
I_0 \cup H_0 = \{s(k + 1) - (k + 1), s(k + 1) - 2(k + 1), \ldots, 2(k + 1), k + 1\}
= \{s(k + 1) - i \mid i = k + 1, 2(k + 1), \ldots, (s - 1)(k + 1)\}.
\]
Recall that this is the set missing from $T_0$ when viewing $T_0$ as an incomplete interval. Thus, $T_0 \cup I_0 \cup H_0$, the entire set of edge labels, is the complete interval $\{s(k + 1) - i \mid i = 1, \ldots, s(k + 1) - 1\}$. Notice that there are exactly $s(k + 1) - 1$ unique integers in this set. Since each of these labels appears once on an edge, and there are $s(k + 1) - 1$ edges, each label must appear exactly once.

Therefore, since the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(G)$ are all distinct, $G$ is graceful.

\[\square\]

2.2 \(\alpha\)-labeling

In this section, we will show that the graceful labeling produced above is, in fact, an \(\alpha\)-labeling. Recall that for a graph $G$, a function $f : V(G) \rightarrow \{0, \ldots, |E(G)|\}$ is an \(\alpha\)-labeling of $G$ if $f$ is a graceful labeling of $G$ with the additional property that there exists some integer $\lambda$ such that for all $uv \in E(G)$, $f(u) \leq \lambda$ and $f(v) > \lambda$. Equivalently, we can say that a graceful labeling of a bipartite graph with bipartition $(A, B)$ is an \(\alpha\)-labeling if there exists an integer $\lambda$ such that $f(a) \leq \lambda$ for every $a \in A$ and $f(b) > \lambda$ for every $b \in B$. We use the latter definition to prove the following theorem.

**Theorem 2.2.** The graceful labeling presented in Section 2.1 is an \(\alpha\)-labeling.

**Proof.** Let $G$ be a uniform $k$-distant tree with $n$ vertices, and let sets $A$ and $B$ be defined as follows:

- $A = \{v_i \mid i \text{ is even}\}$
- $B = \{v_i \mid i \text{ is odd}\}$

Note that both $A$ and $B$ are sets of vertices, but we will also refer to $A$ and $B$ as sets of indices where the indices are taken from the subscripts on the elements of $A$ and $B$. Clearly, $A$ and $B$ are disjoint. To confirm that $G$ is bipartite with bipartition $(A, B)$, we will show that $A$ and $B$ are independent sets. Consider a vertex $v_i$. We have two cases: $v_i$ is either a tail vertex or a spine vertex. If $v_i$ is in the tail, by our naming procedure, we know that $N(v_i)$, the neighbor set of $v_i$, is a subset of $\{v_{i-1}, v_{i+1}\}$. If $v_i$ is in the spine, $N(v_i) \subseteq \{v_{i-1}, v_{i+1}, v_{i+(2k+1)}, v_{i-(2k+1)}\}$. Notice that in any case, for any neighbor $v_j$ of $v_i$, $j$ is of the opposite parity of $i$. Thus, $A$ and $B$ must be independent sets. So, $(A, B)$ forms a valid bipartition of $G$.

Notice that $f$ is strictly increasing as the subscripts of the elements of $A$ increase. Borrowing from calculus, we know then that $\max \{f(v_i) \mid v_i \in A\}$ occurs when $i$ is largest in $A$. If $n$ is even, $i$ is largest when $i = n - 2$, and when $n$ is odd, $i$ is largest at $n - 1$. So, $f(a) \leq f(v_{n-2}) = \frac{n-2}{2}$ for every $a \in A$ when $n$ is even, and $f(a) \leq f(v_{n-1}) = \frac{n-1}{2}$ for all $a \in A$ when $n$ is odd.

Similarly, it is evident that $f$ is strictly decreasing as the subscripts of the elements of $B$ increase. So, we can conclude that $\min \{f(v_i) \mid v_i \in B\}$ occurs when $i$ is largest. For $n$ even, the largest value of $i$ for $v_i$ in $B$ is $n - 1$, and hence, $f(b) \geq n - \frac{n}{2}$ for every $b \in B$. For $n$ odd, the largest value of $i$ is $n - 2$, so $f(b) \geq n - \frac{n-1}{2}$ for every $b \in B$. 

8
Thus, when $n$ is even, $f(a) \leq \frac{n-2}{2}$ for every $a \in A$, and $f(b) > \frac{n-2}{2}$ for every $b \in B$. When $n$ is odd, $f(a) \leq \frac{n-1}{2}$ for every $a \in A$, and $f(b) > \frac{n-1}{2}$ for every $b \in B$. Therefore, our labeling is an $\alpha$-labeling. \qed

3 Appending $a$-stars

To extend the previous result, for a graph $G$ and positive integer $a$ define $app_a(G)$ to be the graph obtained from appending $a$ leaves to each leaf in $G$. In the following sections, we show that, if $G$ is a uniform $t$-distant tree, $app_a(G)$ admits an $\alpha$-labeling.

![Figure 7: $app_a(G)$, where $a = 2$ and $G$ is a uniform 2-distant tree.](image)

3.1 Graceful Labeling

Let $G$ be a uniform $t$-distant tree. To prove that $app_a(G)$ admits an $\alpha$-labeling, we first establish that $app_a(G)$ is graceful. As when producing the function for uniform $k$-distant trees, we will name the vertices of $app_a(G)$ in a specific way before defining the function.

1. Arrange the vertices of $app_a(G)$ so that the tails extend downward from the spine. Notice that each leaf of $G$ combined with the newly appended leaves produces an $a$-star.

2. To begin, label the vertex in the upper left-hand corner $v_0$. Moving down the tail, label the next vertex $v_1$, the next $v_2$, etc., until the end of the first tail of $G$, i.e. $v_t$.

3. Continue naming the first leaf of the first tail with $v_t+1$. Proceed to name all leaves of the first tail, through $v_{t+a}$.

4. Continue naming the leaf directly to the right (on the next tail) with $v_{t+a+1}$, then the next leaf $v_{t+a+2}$, etc., until all leaves on this tail have been named, i.e. until a vertex has been named $v_{t+2a}$.

5. Proceed naming with $v_{t+2a+1}$ on the adjacent tail vertex of $G$. Continue in this manner up the tail toward the spine.
6. After reaching the spine, move to the vertex directly to the right (on the spine), and continue down the next tail.

7. Proceed in this way until all vertices are named. The figure below exemplifies the procedure.

![Diagram of a uniform 2-distant tree]

**Figure 8:** Naming the vertices of $app_3(G)$, where $G$ is a uniform 2-distant tree.

To assign labels to the vertices, we define a function $f$ from the $n$ vertices of the graph to the set \{0, 1, ..., $n - 1$\}, where $n - 1$ is the number of edges. The function $f$ is the same as that in Section 2.1 in all cases for vertices on the $t$-distant tree portion of the graph. If $v_i$ is not a leaf, then

$$f(v_i) = \begin{cases} 
\frac{i}{2} & \text{if } i \text{ is even} \\
\frac{n - i + 1}{2} & \text{if } i \text{ is odd}
\end{cases}$$

However, to label the leaves, we split $f$ into two cases. Notice that if $v_i$ is a leaf, then $i$ is contained in a closed interval of the form $[mt + (m - 1)a + m, mt + ma + (m - 1)]$ or $[mt + ma + m, mt + (m + 1)a + (m - 1)]$, where $m$ is a positive, odd integer. We will refer to these intervals as Form 1 and Form 2, respectively. Let $j = mt + (m - 1)a + m$, and let $l = mt + (m + 1)a + (m - 1)$. If $v_i$ is a leaf, then:

**Case 1:** $t$ is even.

$$f(v_i) = \begin{cases} 
n - (i - \frac{j}{2}) & \text{if } i \text{ is contained in an interval of Form 1} \\
i - \frac{l}{2} & \text{if } i \text{ is contained in an interval of Form 2}
\end{cases}$$
Case 2: $t$ is odd.

$$f(v_i) = \begin{cases} 
    i - \frac{i}{2} & \text{if } i \text{ is contained in an interval of Form 1} \\
    n - (i - \frac{i-1}{2}) & \text{if } i \text{ is contained in an interval of Form 2}
\end{cases}$$

This function is best illustrated using an example.

![A graceful labeling of $app_3(G)$, where $G$ is a uniform 2-distant tree.](image)

We now prove that the function above is a graceful labeling for $app_a(G)$, where $G$ is a $t$-distant tree.

**Theorem 3.1.** Let $a$ and $t$ be positive integers. For every uniform $t$-distant tree $G$, $app_a(G)$ is graceful.

**Proof.** Let $a$ and $t$ be positive integers and $G$ be a uniform $t$-distant tree. Let $s$ be the number of vertices in the spine. Let $n = s(t + a + 1)$. Notice that $n$ is the number of vertices of $app_a(G)$.

Let $f : V(app_a(G)) \to \{0, 1, \ldots, n-1\}$, where $n-1$ is the number of edges of $app_a(G)$, be defined as in the function above. As in Section 2.1, we need to show that the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(app_a(G))$ are all distinct. We will use an approach similar to that of the proof of Theorem 2.1.

$E(app_a(G))$ can be viewed as the union of three sets of edges: edges in the tails of the $t$-distant tree, edges of the $a$-stars, and edges on the spine. We will let $T$ denote the set of edges in the tails, $I$ denote the set of edges in the stars, and $H$ denote the set of edges in the spine. So, $E(app_a(G)) = T \cup I \cup H$.

Once again, we will determine what edge labels appear in $app_a(G)$ by looking at the edge labels appearing in each set $T$, $I$, and $H$ individually. It is beneficial to examine these sets.
in two cases. We begin with the case that $s$ is even. In this case,

$$T = \{ v_0 v_1, v_1 v_2, \ldots, v_{t-1} v_t \} \cup \{ v_{t+2a+1} v_{t+2a+2}, v_{t+2a+2} v_{t+2a+3}, \ldots, v_{t+2a+2} v_{t+2a+1} \}
\cup \{ v_{2t+2a+2} v_{2t+2a+3}, v_{2t+2a+3} v_{2t+2a+4}, \ldots, v_{2t+2a+2} v_{2t+2a+1} \}
\cup \{ v_{3t+2a+3} v_{3t+2a+4}, v_{3t+2a+4} v_{3t+2a+5}, \ldots, v_{3t+2a+2} v_{3t+2a+3} \} \cup \ldots
\cup \{ v_{(s-1)t+sa+(s-1)} v_{(s-1)t+sa+s}, v_{(s-1)t+sa+s} v_{(s-1)t+sa+(s+1)}, \ldots, v_{(s-t+a+1)-2} v_{(s-t+a+1)-1} \}.$$

Let $T_0$ be the set of induced edge labels given by $|f(u) - f(v)|$ for all $uv \in T$. When $s$ is even, there are two cases of $T_0$.

**Case 1:** $t$ is even.

$$T_0 = \{ |0 - (n - 1)|, \ldots, |n - \frac{t}{2} - \frac{t}{2}| \} \cup \{ |n - \frac{t+2a+2}{2} - \frac{t+2a+2}{2}|, \ldots, |n - \frac{2t+2a+2}{2} - \frac{2t+2a+2}{2}| \} \cup \{ |n - \frac{2t+2a+2}{2} - \frac{2t+2a+4}{2}|, \ldots, |n - \frac{3t+2a+2}{2} - \frac{3t+2a+2}{2}| \} \cup \ldots \cup \{ |n - \frac{(s-1)t+sa+1}{2} - \frac{(s-1)t+sa+1}{2}|, \ldots, |n - \frac{s(t+a+1)-2}{2} - \frac{s(t+a+1)-2}{2}| \}$$

$$= \{ |1 - n|, \ldots, |n - t| \} \cup \{ |n - (t + 2a + 2)|, \ldots, |(2t + 2a + 1) - n| \} \cup \ldots \cup \{ |t|, \ldots, |1| \}.$$

**Case 2:** $t$ is odd.

$$T_0 = \{ |0 - (n - 1)|, \ldots, |n - \frac{t}{2} - \frac{t+1}{2}| \} \cup \{ |n - \frac{t+2a+1}{2} - \frac{t+2a+3}{2}|, \ldots, |n - \frac{2t+2a+2}{2} - \frac{2t+2a+4}{2}|, \ldots, |n - \frac{3t+2a+3}{2} - \frac{3t+2a+3}{2}| \} \cup \ldots \cup \{ |n - \frac{(s-1)t+sa+1}{2} - \frac{(s-1)t+sa+1}{2}|, \ldots, |n - \frac{s(t+a+1)-2}{2} - \frac{s(t+a+1)-2}{2}| \}$$

$$= \{ |1 - n|, \ldots, |t - n| \} \cup \{ |t + 2a + 2 - n|, \ldots, |(2t + 2a + 1) - n| \} \cup \ldots \cup \{ |t|, \ldots, |1| \}.$$

Notice that upon further simplification, both of these cases of $T_0$ result in the same set. So, for any $s$ even,

$$T_0 = \{ n-1, \ldots, n-t \} \cup \{ n - (t + 2a + 2), \ldots, n - (2t + 2a + 1) \}
\cup \{ n - (2t + 2a + 3), \ldots, n - (3t + 2a + 2) \} \cup \ldots \cup \{ t, \ldots, 1 \}.$$

Recall that we defined $I$ to be the the set of edges in the stars. When $s$ is even,

$$I = \{ v_t v_{t+1}, v_t v_{t+2}, \ldots, v_t v_{t+a} \} \cup \{ v_{t+2a+1} v_{t+2a+2}, v_{t+2a+2} v_{t+2a+3}, \ldots, v_{t+2a+1} v_{t+2a} \}
\cup \{ v_{3t+2a+2} v_{3t+2a+3}, v_{3t+2a+3} v_{3t+2a+4}, \ldots, v_{3t+2a+2} v_{3t+2a+1} \} \cup \ldots
\cup \{ v_{(s-1)t+sa+(s-1)} v_{(s-1)t+(s-1)a+(s-1)}, v_{(s-1)t+(s-1)a+(s-1)} v_{(s-1)t+(s-1)a+(s-2)}, \ldots,
\}$

$$v_{(s-1)t+sa+(s-1)} v_{(s-1)t+sa+(s-2)} \}.$$

Let $I_0$ be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in I$. As with $T_0$, there are initially two cases of $I_0$. 

12
Case 1: \( t \) is even.

\[
I_0 = \{ \left\lfloor \frac{n}{2} \right\rfloor - (t + 1 - t) \}, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - (t + a - t) \} \cup \{ \left\lfloor \frac{n - \left\lfloor \frac{t+2a+2}{2} \right\rfloor}{2} \right\rfloor - (t + a + 1 - \frac{t+2a}{2}) \}, \ldots, \\
\left\lfloor \frac{n - \left\lfloor \frac{t+2a+2}{2} \right\rfloor}{2} \right\rfloor - (t + 2a - \frac{t+2a}{2}) \} \cup \{ \left\lfloor \frac{3t+2a+2}{2} \right\rfloor - (n - (3t + 2a + 3 - \frac{3t+2a+2}{2})) \}, \ldots, \\
\left\lfloor \frac{3t+2a+2}{2} \right\rfloor - (n - (3t + 3a + 2 - \frac{3t+2a+2}{2})) \} \cup \ldots \cup \\
\{ \left\lfloor n - \left\lfloor \frac{(s-1)t+sa+s}{2} \right\rfloor \right\rfloor - ((s-1)t + (s-1)a + (s-1) - \frac{(s-1)t+sa+(s-2)}{2}) \}, \ldots, \\
\left\lfloor n - \left\lfloor \frac{(s-1)t+sa+s}{2} \right\rfloor \right\rfloor - ((s-1)t + sa + (s-2) - \frac{(s-1)t+sa+(s-2)}{2}) \} \}
\]

\[
= \{(t+1-n), \ldots, (t+a-n)\} \cup \{n-(t+a+2), \ldots, n-(t+2a+1)\} \cup \{(3t+2a+3)-n, \ldots, (3t+3a+2)-n\} \cup \ldots \cup \{t+a, \ldots, t+1\}.
\]

Case 2: \( t \) is odd.

\[
I_0 = \{ \left\lfloor n - \frac{t+1}{2} \right\rfloor - (t + 1 - t) \}, \ldots, \left\lfloor n - \frac{t+1}{2} \right\rfloor - (t + a - t) \} \cup \\
\{ \left\lfloor \frac{t+2a+1}{2} \right\rfloor - (n - (t + a + 1 - t) \}, \ldots, \left\lfloor \frac{t+2a+1}{2} \right\rfloor - (n - (t + 2a - t) \} \cup \\
\{ \left\lfloor n - \left\lfloor \frac{3t+2a+3}{2} \right\rfloor \right\rfloor - [3t + 2a + 3 - \frac{3t+2a+3}{2}] \}, \ldots, \left\lfloor n - \left\lfloor \frac{3t+2a+3}{2} \right\rfloor \right\rfloor - [3t + 3a + 2 - \frac{3t+2a+3}{2}] \} \cup \\
\ldots \cup \{ \left\lfloor \frac{(s-1)t+sa+(s-1)}{2} \right\rfloor - n - ((s-1)t + (s-1)a + (s-1) - \frac{(s-1)t+sa+(s-2)-1}{2}) \}, \\
\left\lfloor \frac{(s-1)t+sa+(s-1)}{2} \right\rfloor - n - ((s-1)t + sa + (s-2) - \frac{(s-1)t+sa+(s-2)-1}{2}) \} \}
\]

\[
= \{n-(t+1), \ldots, n-(t+a)\} \cup \{n-(t+2), \ldots, n-(t+1)\} \cup \{(3t+2a+3), \ldots, n-(3t+3a+2)\} \cup \ldots \cup \{t+a, \ldots, t+1\}.
\]

In both cases, we obtain:

\[
I_0 = \{ n -(t+1), \ldots, n - (t+a) \} \cup \{ n -(t+2), \ldots, n - (t+1) \} \\
\cup \{ n -(3t+2a+3), \ldots, n -(3t+3a+2) \} \cup \ldots \cup \{ t+a, \ldots, t+1 \}.
\]

So, when \( s \) is even, we have that

\[
T_0 \cup I_0 = \{ n -1, \ldots, n - (t+a) \} \cup \{ n -(t+a+2), \ldots, n -(2t+2a+1) \} \\
\cup \{ n -(2t+2a+3), \ldots, n -(3t+3a+2) \} \cup \ldots \cup \{ t+a, \ldots, 1 \} \\
= \{ n - i \mid i = 1, \ldots, t+a \} \cup \{ n - i \mid i = t+a+1, \ldots, 2t+2a+1 \} \\
\cup \{ n - i \mid i = 2t+2a+3, \ldots, 3t+3a+2 \} \cup \ldots \\
\cup \{ n - i \mid i = (s-1)t + (s-1)a + s, \ldots, (s-1)a + s \} \\
= \{ n - i \mid i = 1, \ldots, n-1 \} \setminus \{ n - i \mid i = t+a+1, 2(t+a+1), \ldots, (s-1)(t+a+1) \}
\]

When \( s \) is odd, this union takes shape a little differently. When \( s \) is odd,

\[
T = \{ v_0, v_1, v_2, \ldots, v_{t-1} \} \cup \{ vt_{2a+1}, vt_{2a+2}, vt_{2a+3}, \ldots, v_{2t+2a}v_{2t+2a+1} \} \\
\cup \{ vt_{2a+2}, vt_{2a+3}, \ldots, v_{3t+2a+1}v_{3t+2a+2} \} \\
\cup \{ vt_{3t+4a+3}, vt_{3t+4a+4}, \ldots, v_{4t+4a+2}v_{4t+4a+3} \} \cup \ldots \\
\cup \{ vt_{(s-1)t+(s-1)a+(s-1)}v_{(s-1)t+(s-1)a+(s-1)+1}, \ldots, v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+(s-1)+1} \}.
\]
As when \( s \) is even, we have two cases of \( T_0 \) when \( s \) is odd.

**Case 1:** \( t \) is even.

\[
T_0 = \{0 - \frac{n-1}{2}, \ldots, \left[ n - \frac{t-1}{2} \right] \} \cup \{ \left[ n - \frac{t+2a+2}{2} \right] - \frac{t+2a}{2}, \ldots, \left[ 2n - \frac{t+2a+2}{2} \right] - \frac{t+2a+4}{2} \} \cup \left[ \frac{t+2a}{2} - \frac{t+2a+2}{2} \right]
\]

\[
= \{1 - n, \ldots, |n - t|\} \cup \{ \left[ n - (t + 2a + 2) \right], \ldots, \left[ 2(t + 2a + 3) - n \right] \} \cup \{ \left[ n - (3t + 2a + 2) \right] \} \cup \ldots \cup \{ \left[ -(t + a) \right], \ldots, |a + 1| \}.
\]

**Case 2:** \( t \) is odd.

\[
T_0 = \{0 - \frac{n-1}{2}, \ldots, \left[ - \frac{t-1}{2} \right] \} \cup \{ \left[ n - \frac{t+1}{2} \right] \} \cup \{ \left[ n - \frac{t+2a+3}{2} \right] - \frac{t+2a}{2}, \ldots, \left[ 2n - \frac{t+2a+2}{2} \right] - \frac{t+2a+4}{2} \} \cup \left[ \frac{t+1}{2} - \frac{t+2a+2}{2} \right]
\]

\[
= \{1 - n, \ldots, |t - n|\} \cup \{ \left[ t + 2a + 2 - n \right], \ldots, \left[ 2(t + 2a + 3) - n \right] \} \cup \{ \left[ t + 2a + 2 - n \right] \} \cup \ldots \cup \{ |t + a|, \ldots, -(a + 1)| \}.
\]

Again, these reduce to the same set:

\[
T_0 = \{ n - 1, \ldots, n - t \} \cup \{ n - (t + 2a + 2), \ldots, n - (2t + 2a + 1) \} \cup \{ n - (2t + 2a + 3), \ldots, n - (3t + 2a + 2) \} \cup \ldots \cup \{ t + a, \ldots, a + 1 \}.
\]

When \( s \) is odd,

\[
I = \{ v_1v_{t+1}, v_1v_{t+2}, \ldots, v_tv_{t+a} \} \cup \{ v_{t+2a+1}v_{t+a+1}, v_{t+2a+1}v_{t+a+2}, \ldots, v_{t+2a+1}v_{t+a} \}
\]

\[
\cup \{ v_{3t+2a+2}v_{3t+2a+3}, v_{3t+2a+2}v_{3t+2a+4}, \ldots, v_{3t+2a+2}v_{3t+2a+3} \} \cup \ldots
\]

\[
\cup \{ v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+s}, v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+s+(s+1)}, \ldots, v_{st+(s-1)a+(s-1)}v_{st+(s-1)a+(s+1)} \}.
\]

Not surprisingly, we again have two cases of \( I_0 \).

**Case 1:** \( t \) is even.

\[
I_0 = \{ \left[ \frac{1}{2} - \frac{n - (t + 1 - \frac{1}{2})}{2} \right], \ldots, \left[ \frac{1}{2} - \frac{n - (t + a - \frac{1}{2})}{2} \right] \} \cup \{ \left[ n - \frac{t+2a+2}{2} \right] - \frac{t+2a}{2}, \ldots, \left[ 2n - \frac{t+2a+2}{2} \right] - \frac{t+2a+4}{2} \}
\]

\[
\cup \left[ \frac{t+2a}{2} - \frac{t+2a+2}{2} \right]
\]

\[
= \{ |t+1-n|, \ldots, |t+a-n| \} \cup \{ |n-(t+a+2)|, \ldots, |n-(t+2a+1)| \} \cup \{ |3t+2a+3-n|, \ldots, |3t+3a+2| \} \cup \ldots \cup \{ |a|, \ldots, |1| \}.
\]

\]

\]

\]
Case 2: $t$ is odd.

\[
I_0 = \{[n - \frac{t+1}{2}] - (t + 1 - \frac{t+1}{2})], \ldots, [n - \frac{t+1}{2}] - (t + a - \frac{t+1}{2})] \cup \\
\{[\frac{t+2a+1}{2}] - [n - (t + a + 1 - \frac{t+2a-1}{2})], \ldots, \frac{t+2a+1}{2} - [n - (t + 2a - \frac{t+2a-1}{2})] \} \cup \\
\{[n - \frac{3t+2a+3}{2}] - [3t + 2a + 3 - \frac{3t+2a+3}{2}], \ldots, [n - \frac{3t+2a+3}{2}] - [3t + 3a + 2 - \frac{3t+2a+3}{2}] \} \\
\cup \ldots \cup \{[n - \frac{st+(s-1)a+s}{2}] - [st + (s-1)a + s - \frac{st+(s-1)a+s}{2}], \ldots, \\
[n - \frac{st+(s-1)a+s}{2}] - [s(t + a + 1) - 1 - \frac{st+(s-1)a+s}{2}]\} \\
= \{[n - (t+1)], \ldots, [n - (t+a)] \} \cup \{(t+a+2) - n, \ldots, (t+2a+1) - n \} \cup \{n - (3t+2a+3), \ldots, n - (3t + 3a + 2) \} \cup \ldots \cup \{a, \ldots, 1 \}.
\]

So, when $s$ is odd,

\[
I_0 = \{n - (t+1), \ldots, n - (t+a) \} \cup \{n - (t+a+2), \ldots, n - (t+2a+1) \} \\
\cup \{n - (3t+2a+3), \ldots, n - (3t+3a+2) \} \cup \ldots \cup \{a, \ldots, 1 \}.
\]

Notice that the union of $T_0$ and $I_0$ is the same as when $s$ is even, i.e.

\[
T_0 \cup I_0 = \{n - 1, \ldots, n - (t+a) \} \cup \{n - (t+a+2), \ldots, n - (2t+2a+1) \} \\
\cup \{n - (2t+2a+3), \ldots, n - (3t+3a+2) \} \cup \ldots \cup \{t+a, \ldots, 1 \} \\
= \{n - i \mid i = 1, \ldots, t+a \} \cup \{n - i \mid i = t+a+2, \ldots, 2t+2a+1 \} \\
\cup \{n - i \mid i = 2t+2a+3, \ldots, 3t+3a+2 \} \cup \ldots \\
\cup \{n - i \mid i = (s-1)t + (s-1)a + s, \ldots, s(t + a + 1) - 1 \} \\
= \{n - i \mid i = 1, \ldots, n-1 \} \setminus \{n - i \mid i = t+a+1, 2(t+a+1), \ldots, (s-1)(t+a+1) \}.
\]

If we view this union as an interval, note that we are missing the set \{n - i \mid i = t+a+1, 2(t+a+1), \ldots, (s-1)(t+a+1) \}. Recall that we named the set of edges in the spine $H$. Although they are similar, we will split $H$ into two cases.

Case 1: $s$ is even.

\[
H = \{v_0v_{2t+2a+1}, v_{2t+2a+1}v_{2t+2a+2}, \ldots, v_{(s-2)(t+a+1)}v_{s(t+a+1)-1} \}.
\]

Case 2: $s$ is odd.

\[
H = \{v_0v_{2t+2a+1}, v_{2t+2a+1}v_{2t+2a+2}, \ldots, v_{(s-1)(t+a+1)-1}v_{s(t+a+1)-1} \}.
\]

Letting $H_0$ be the set of induced edge labels $|f(u) - f(v)|$ for all $uv \in H$, we have:

Case 1: $s$ is even.

\[
H_0 = \{[0 - [n - \frac{2t+2a+2}{2}]], [n - \frac{2t+2a+2}{2}] - \frac{2t+2a+2}{2}, \ldots, \frac{(s-2)(t+a+1)}{2} - [n - \frac{s(t+a+1)}{2}] \} \\
= \{(t + a + 1) - n, [n - 2(t + a + 1)], \ldots, -(t + a + 1) \}.
\]

Case 2: $s$ is odd.

\[
H_0 = \{[0 - [n - \frac{2t+2a+2}{2}]], [n - \frac{2t+2a+2}{2}] - \frac{2t+2a+2}{2}, \ldots, [n - \frac{(s-1)(t+a+1)}{2}] - \frac{(s-1)(t+a+1)}{2} \} \\
= \{(t + a + 1) - n, [n - 2(t + a + 1)], \ldots, t + a + 1 \}.
\]
Again, these reduce to the same set:

$$H_0 = \{n - (t + a + 1), n - 2(t + a + 1), \ldots, t + a + 1\} = \{n - i \mid i = t + a + 1, 2(t + a + 1), \ldots, (s - 1)(t + a + 1)\}.$$ 

Recall that this is exactly the set we are missing from $T_0 \cup I_0$ when viewing it as an incomplete interval. Thus, $T_0 \cup I_0 \cup H_0$, the entire set of edge labels, is the complete interval \{n - i \mid i = 1, \ldots, s(t + a + 1) - 1\}. So, there are exactly $s(t + a + 1) - 1 = n - 1$ unique integers in this interval, and each of these labels appears once on an edge. Since there are $n - 1$ edges, we know that each edge label must appear exactly once. Therefore, since the induced edge labels given by $|f(u) - f(v)|$ for every $uv \in E(app_\alpha(G))$ are all distinct, $app_\alpha(G)$ is graceful. 

\[\square\]

### 3.2 $\alpha$-labeling

This labeling, like the labeling from Section 2, is also an $\alpha$-labeling. We show this in a similar manner.

**Theorem 3.2.** The graceful labeling presented in Section 3.1 is an $\alpha$-labeling.

**Proof.** Let $a$ and $t$ be positive integers and $G$ be a uniform $t$-distant tree. By saying a vertex $v$ is in $G$, we mean $v \in V(G)$. Note that any vertex in $app_\alpha(G)$ is either in $G$, or it is a leaf. We will present this proof in two cases:

**Case 1: $t$ is even**

Define sets $A$ and $B$ as follows:

$A = \{v_i \mid v_i \in G \text{ and } i \text{ is even}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 2}\}$

$B = \{v_i \mid v_i \in G \text{ and } i \text{ is odd}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 1}\}$

Note that both $A$ and $B$ are sets of vertices, but we will also refer to $A$ and $B$ as sets of indices where the indices are taken from the subscripts on the elements of $A$ and $B$. We can see that $A$ and $B$ are disjoint. To be sure $app_\alpha(G)$ is bipartite with bipartition $(A, B)$, we must show that $A$ and $B$ are independent. Consider a vertex $v_i \in A$. This vertex must satisfy one of three cases: $v_i$ could be in $G$ but not adjacent to a leaf, $v_i$ could be in $G$ and adjacent to a leaf, or it could be a leaf. If $v_i$ is in $G$ but not adjacent to a leaf, then by our naming scheme, $N(v_i)$, the neighbor set of $v_i$, is a subset of \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}. In this case, any neighbor $v_j$ of $v_i$ would be in $B$, since $j$ must be in $G$ and is of opposite parity than $i$.

If $v_i$ is in $G$ and is adjacent to a leaf, then $N(v_i) = \{v_{i-1}\} \cup \{v_{i+1}, \ldots, v_{i+a}\}$, where $v_{i-1} \in G$. Note that according to our naming procedure, $i$ is of the form $mt + (m - 1)a + (m - 1)$ for some odd integer $m$. Thus, $N(v_i) = \{v_{i-1}\} \cup \{v_{mt+(m-1)a+m}, \ldots, v_{mt+ma+(m-1)}\}$. Thus, for any neighbor $v_j$ of $v_i$, either $j$ is in $G$ and is of opposite parity than $i$, or $j$ is in an interval of Form 1. Thus, $v_j$ must be in $B$.

Finally, consider $v_i$, where $v_i$ is a leaf. Since $v_i \in A$, $i$ must be contained in an interval of Form 2, i.e. $i \in \{mt + ma + m, \ldots, mt + (m + 1)a + (m - 1)\}$ for some positive, odd integer $m$. Note also that for each unique $m$, $N(v_i) = \{v_{mt+(m+1)a+m}\}$ for every $i \in \{mt + ma + m, \ldots, mt + (m + 1)a + (m - 1)\}$.
conclude that max $\{v \mid v \in A, A \text{ is a leaf and } i \text{ is contained in an interval of Form 1}\}$ occur when $s$ is largest. When $s$ is even, the largest value of $i$ in $A$ is $n-2$, where $v_{n-2} \in G$. So, for every $x \in A$, $f(x) \leq \frac{n-2}{2}$. When $s$ is odd, $i$ is largest in $A$ where $i = n-(a+1)$, where $v_i$ is in $G$. Thus, $f(x) \leq \frac{n-(a+1)}{2}$ for all $x \in A$.

On the other hand, $f(b)$ is decreasing as the subscripts in $B$ increase; hence, min $\{f(v_i) \mid v_i \in B\}$ occurs where $i$ is largest in $B$. When $s$ is even, the largest subscript found in $B$ is $n-1$, where $v_{n-1} \in G$. Thus, for all $b \in B$, $f(b) \geq n - \frac{n}{2} = n - \frac{n}{2} > \frac{n-2}{2}$. When $s$ is odd, $i$ is largest in $B$ where $i = n-1$, where $v_i$ is a leaf. Therefore, when $s$ is odd, $f(b) \geq f(v_{n-1}) = \frac{n-(a-1)}{2} > \frac{n-(a+1)}{2}$ for every $b \in B$.

**Case 2:** $t$ is odd

$A = \{v_i \mid v_i \in G \text{ and } i \text{ is even}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 1}\}$

$B = \{v_i \mid v_i \in G \text{ and } i \text{ is odd}\} \cup \{v_i \mid v_i \text{ is a leaf and } i \text{ is contained in an interval of Form 2}\}$

Again, note that both $A$ and $B$ are sets of vertices, but we will also refer to $A$ and $B$ as sets of indices where the indices are taken from the subscripts on the elements of $A$ and $B$. While it is clear that $A$ and $B$ are disjoint, we must show that they are independent to confirm that $\text{app}_a(G)$ is bipartite with bipartition $(A, B)$. Consider a vertex $v_i \in A$. If $v_i$ is in $G$ but not adjacent to a leaf, we know by our naming scheme that $N(v_i) \subseteq \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$. So, any vertex adjacent to $v_i$ would have a subscript of opposite parity as $i$ and would thus be in $B$. If $v_i$ is in $G$ but adjacent to a leaf, according to our naming procedure, $i$ must be of the form $mt + (m+1)a + m$ for some positive, odd integer $m$. Thus, $N(v_i) = \{v_{i+1}, v_{i-1}, \ldots, v_{i+m}\}$, so any neighbor of $v_i$ must be in $B$. If $v_i$ is a leaf, then $i \in \{mt + (m-1)a + m, \ldots, mt + ma + (m-1)\}$ for some positive, odd integer $m$, and $N(v_i) = \{v_{mt+(m-1)a+(m-1)}\}$. Thus, it is clear that $N(v_i) \subseteq B$ when $v_i \in A$.

Now let $v_i \in B$. If $v_i \in G$ and is not adjacent to a leaf, again we have that $N(v_i) \subseteq \{v_{i+(2t+2a+1)}, v_{i-(2t+2a+1)}, v_{i+1}, v_{i-1}\}$, so any neighbor of $v_i$ would be in $A$. If $v_i$ is in $G$ and adjacent to a leaf, then $N(v_i) = \{v_{i-1}, \ldots, v_{i+a}\}$, where $v_{i-1} \in G$. Since, by our naming procedure, $i$ is of the form $mt + (m-1)a + (m-1)$, it’s easy to see that any neighbor of $v_i$ would be in $A$. If $v_i$ is a leaf, then $i \in \{mt + ma + m, \ldots, mt + ma + (m-1)\}$ for some positive, odd integer $m$, and $N(v_i) = v_{mt+(m-1)a+m} \in G$. Thus, since $mt + (m+1)a + m$ is even, we know that $N(v_i) \subseteq A$. Therefore, $A$ and $B$ are independent sets of vertices.

Notice that $f$ is increasing as the subscripts of the elements of $A$ increase. Again, we can conclude that max $\{f(v_i) \mid v_i \in A\}$ occurs when $i$ is largest. When $s$ is even, $i$ is largest when
$i = n - 2$, where $v_i \in G$. When $s$ is odd, $i$ is largest at $n - 1$, where $v_i$ is a leaf. So, $f(x) \leq \frac{n-2}{2}$ for every $x \in A$ when $s$ is even and $f(x) \leq s(t + a + 1) - \left(\frac{st+1+(s-1)a+(s-1)}{2}\right) = \frac{n+(a-2)}{2}$ for every $x \in A$ when $s$ is odd.

Similarly, $f$ is decreasing as subscripts of the elements of $B$ decrease. Thus, we know that $\min\{f(v_i) \mid v_i \in B\}$ occurs when $i$ is largest. For $s$ even, the largest value of $i$ for $v_i$ in $B$ is $n - 1$, where $v_i$ is in $G$, so $f(b) \geq n - \frac{n}{2} > \frac{n-2}{2}$ for every $b \in B$. When $s$ is odd, the largest valued subscript in $B$ is $i = n - (a + 1)$, where $v_i$ is in $G$. Thus, $f(b) \geq n - \frac{n-(a+1)+1}{2} = \frac{n+a}{2} > \frac{n+(a-2)}{2}$ for every $b \in B$.

We have shown that in every case, for bipartition $(A, B)$ of $app_a(G)$, there exists an integer $\lambda$ such that $f(x) \leq \lambda$ for every $x \in A$ and $f(b) > \lambda$ for every $b \in B$. Therefore, our labeling is an $\alpha$-labeling.

\qed

4 Concluding Remarks

Our labeling method has been successful for obtaining harmonious, graceful, and $\alpha$-labelings of uniform $k$-distant trees, as well as graceful and $\alpha$-labelings of those same graphs with stars appended to the ends of the tails. We hope that the same method can be used to show that all uniform $k$-distant trees with stars appended are harmonious. It is noted by Gallian in [5] that whether or not lobsters are harmonious seems to have attracted no attention thus far. It would be interesting to see if a similar approach works for showing that all lobsters are harmonious. Another problem to consider is to allow for different numbers of leaves to be appended to each tail of a uniform $k$-distant tree, and to find an $\alpha$-labeling for this class of graphs.

APPENDIX

The following definitions may be useful to reference. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

Two vertices are \textit{adjacent} if they are connected by an edge.

A \textit{path} in $G$ is a sequence of distinct vertices such that there is an edge between each set of consecutive vertices.

A \textit{cycle} is a path that begins and ends with the same vertex. If $G$ has no cycles, then we call it an \textit{acyclic} graph.

$G$ is \textit{connected} if and only if every pair of vertices in $V(G)$ can be joined by a path in $G$.

A \textit{tree} is a connected, acyclic graph.

The \textit{degree} of a vertex is the number of edges incident to that vertex. A vertex of degree one is called a \textit{leaf}.
A set of vertices for which there is no edge with both endpoints in that set is called an independent set of vertices.

$G$ is bipartite with bipartition $(A, B)$ if $V(G)$ can be partitioned into two disjoint subsets $A$ and $B$ such that $A$ and $B$ are each independent sets of vertices.

Let $H$ be a graph with vertex set $V(H)$ and edge set $E(H)$. Graphs $G$ and $H$ are isomorphic (denoted $G \cong H$) if there exists a bijection $f : V(G) \rightarrow V(H)$ such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$. The function $f$ is called an isomorphism. An isomorphism of a graph to itself is called an automorphism.

$C_n$ denotes the cycle on $n$ vertices.

A complete graph is a graph in which every pair of distinct vertices is connected by an edge. $K_n$ denotes the complete graph on $n$ vertices.

$\mathbb{Z}_n$ denotes the group of integers modulo $n$.

References


