Undergraduate Economic Review

Volume 17 | Issue 1 Article 12

2020

A Quantitative Analysis on BitMEX Perpetual Inverse Futures XBTUSD Contract

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Abstract

The perpetual inverse futures contract is a recent and most popularly traded cryptocurrency derivative over crypto derivatives exchanges. Exchanges implement a liquidation mechanism that terminates positions which no longer satisfy maintenance requirements. In this study, we use regression, stochastic calculus, and simulation methods to provide a quantitative description of the wealth/return process for holding an XBTUSD contract on BitMEX, examine the funding rate and index price properties, and relate liquidation to leverage as a stopping time problem. The results will help investors understand liquidation to optimize their trading strategy and researchers in studying the design of crypto derivatives.

Keywords

BitMEX, funding rate, index price, liquidation, leverage

Cover Page Footnote

Acknowledgements: This research is supported by the SURF program of Mellon College of Science, Carnegie Mellon University. Professor Ariel Zetlin-Jones and Professor Bryan Routledge at the Tepper School of Business, Carnegie Mellon University, provided advise for the research. Footnote:

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Erratum

The author wishes to acknowledge and cite the unpublished lecture notes of Professor Dmitry Kamarov, specifically on manuscript pages 17 and 21 under "Procedures." (Full citation: Kramkov, D. (2020). Continuous-Time Finance Course Lecture Notes at Carnegie Mellon University.)

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1 Introduction

1.1 Overview

Bitcoin (Ƀ), introduced by Satoshi Nakamoto, is the first decentralized digital currency based on blockchain technology (Nakamoto 2008). Since the advent of the first block in 2009, Bitcoin has been the leader among digital assets, with the largest market capitalization, and is among one of the highest traded commodities. Associated with Bitcoin's high (and risky) value is high interest in speculating on or hedging against its future price movements which have led to a growing market for Bitcoin-related derivatives. One crypto derivative that started trading in recent years and is quickly dominating several crypto exchanges is the perpetual inverse futures contract. Among the Bitcoin derivatives exchanges offering the product, BitMEX generates the highest daily trading volume. Launched in May 2016, XBTUSD is BitMEX's most popular product, allowing traders to speculate on the Bitcoin / USD exchange rate with up to 100x leverage, which means traders could have a purchasing power of up to 100 times of their own investments.

The perpetual inverse futures contract is novel. Similar to typical futures options, XBTUSD updates profits and losses (P&L) every period (8 hours). In contrast to traditional futures, there is no expiry (maturity) for the perpetual option, and the margin and profit/loss are denominated in the base currency instead of the quote currency. In XBTUSD, the base currency is Bitcoin, and the quote currency is USD (BitMEX, 2020).

Because of the novelty of perpetual inverse futures, there are very few studies of its characteristics. Recent studies have discussed the relationship between the Bitcoin spot market and the futures market (Baur & Dimpfl, 2019) and have shown that BitMEX derivatives have positive net spillover effects on the spot market (Alexander et al., 2020). Nimmagadda and Sasanka studied the BitMEX funding rate, proved its Heteroskedastic nature, and suggested a causal relationship between BitMEX funding rates and the USDXBT contract based on Granger causality (2019). Deng et al. studied the mean-variance tradeoff of Bitcoin inverse futures, calculated the expectation and variance on the contract's returns (2020), and computed an optimal hedging strategy (2020).

Despite the exploration of market influences and trading strategies, there is still a lack of documentation of the cash flows for holding a position in a contract. To estimate the fair present value of a position, whether the contract is underpriced or overpriced, it is essential to understand the cash flows. The BitMEX website provides only basic descriptions about XBTUSD's functioning, without offering systematic analysis or formulas on the wealth process. Deng et al. build a theoretical model for the evolution of the intrinsic value of the futures (2019), yet the model fails to capture the actual realized P&L (similar to dividends) for an investor holding the position. Therefore, in this study, we will fill this gap by giving a comprehensive description of the cash flows that explains the wealth (return) process for holding a long position on XBTUSD.

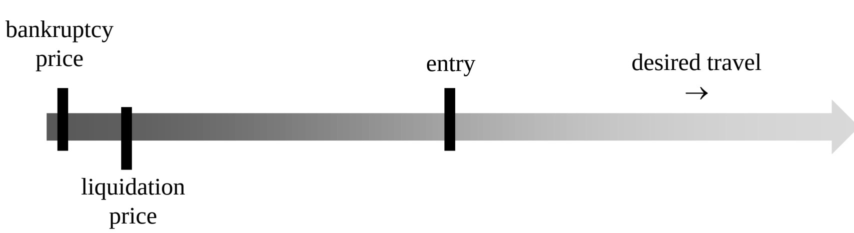


Figure 1. Liquidation Price in Relation to Bankruptcy Price (**Long** Position) (BitMEX Simplified, 2018)

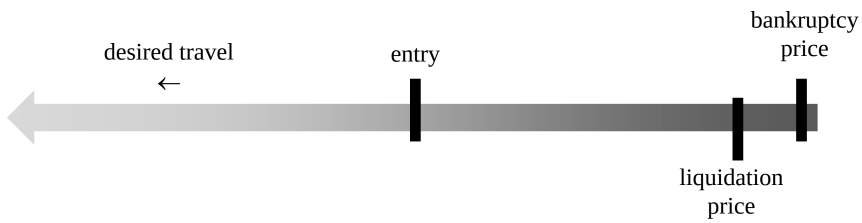


Figure 2. Liquidation Price in Relation to Bankruptcy Price (**Short** Position)

Allowing high leverage of up to 100x amplifies both profits and losses of the investor. To avoid bankruptcy when investors are trading at high leverage, BitMEX implements a liquidation price slightly above (or below, for short positions) the bankruptcy price, as shown in **Figures 1 & 2** (BitMEX Simplified, 2018). When the inverse futures contract price drops to (or rises to, for short positions) the liquidation price of the position, BitMEX will cancel any open orders in that contract in order to free up margin and maintain the position. If the maintenance margin requirement is unsatisfied, the position will be liquidated by the exchange at the bankruptcy price. <u>(Note: Since the inverse futures contract price approximates the index price in most time periods, in this paper, we adopt the process for index price and evaluate liquidation probabilities under the process for index price)</u>

Relevant studies on cryptocurrency exchanges suggest that BitMEX's liquidation mechanism improves the exchange's efficiencies (liquidity and risks) and comes as a risk to the investors. Kao et al. suggest that the incentive for liquidation mechanisms in Ethereum assets' Compound protocol is to ensure the exchange's solvency; they also suggest that in an investor's perspective, increased liquidation incentives suppress borrowing demand (2020). Considering the overall volatility of the underlying asset, or the index price, it is necessary for the investor to restrict maximum leverage (Lee & Cho, 2018).

It is important for investors to understand the relationship between the liquidation limit and leverage so that they can maximize their leverage while keeping a low level of liquidation risk. Since no prior study has examined such a relationship, in this study, we model the liquidation price as a function of leverage and analyze liquidation probability and expected liquidation time of the position. We use the stochastic calculus method to derive formulas for the expected liquidation time and the probability of liquidation before a certain time when ignoring funding and fees and use simulation to characterize these quantities when considering funding and fees. This will help investors evaluate the risks for taking certain leverage and helping develop an optimal trading strategy.

Modeling liquidation risks will also enable us to compare the theoretical value of a contract with the actual traded price. Any significant price difference may reflect potential price manipulations. In a greater context, this research will further the understanding of designing derivatives exchanges.

The next part of the introductory section introduces all the key terminologies and their features used throughout this paper. In the remainder of this paper, Section 2 will offer a quantitative description of holding a long position of the XBTUSD contract and provide a regression model to characterize the funding rate as a function of index price and the financial market's performance.

We study the long position of XBTUSD contract because the short position scenario is similar and that given the liquidation price is less than the initial index price, the long position is uniquely interesting with the potential of never getting liquidated when the index price continuously raises. Next, in Section 3, we analyze the effects of leverage on liquidation for the long position using stochastic calculus methods and simulation. For the scenario without funding and fees, we derive the formulas for expected liquidation time and the probability of getting liquidated before a certain future time, and provided financial interpretations on the results. For the case with funding and fees, we reference relevant constants' values through regressions on historical data, and then use these constants to run simulations on geometric Brownian motion and mean-reverting process models. The data we collect from these simulations are used to build regression models to assess the expected liquidation time and the probability of getting liquidated before a certain future time, and the models are analyzed to provide financial conclusions. Lastly, in Section 4, we summarize the results of this research and discuss some remaining questions.

1.2 Key Terminologies & Features

Table 1. Key Terminologies

Category	Variable	Symbol	Descriptions
Contract & Funding	Funding Rate	(r_t)	Funding is the primary mechanism to tether to spot price. Funding rate is consisted of Interest Rate and the Premium / Discount. Funding amount at time t is $funding\ amount_t$ (Ƀ) $=$ $position\ value_t \times funding\ rate_t$
	Index Price	(S_t)	The BitMEX Index Price .BXBT is calculated as a weighted average of the XBT Last Price for five constituent exchanges: Coinbase (46.98%), Bitstamp (25.73%), Kraken (17.97%), Gemini (6.69%), Bittrex (2.63%). <u>Since the inverse futures contract price approximates the index price in most time periods, we adopt the process for S_t and evaluate liquidation probabilities under the process for S_t</u> A bitcoin is worth Ƀ $1 = \$S_t$; a contract is worth $\$1 =$ Ƀ $\frac{1}{S_t}$
Position	Leverage	L	Borrowings from the exchange that amplify gains and losses of a position. $L \in [1,100]$
	Margin Size	N	It is number of contracts bought with your own money initially. The total number of contracts you own is scaled up by leverage: NL. In the following sections, we assume that it remains the same until liquidation
	Position Value	(P_t)	Independent of leverage, this is equal to the notional amount of the contracts, i.e. marginal amount scaled up by leverage, or "total position size"

			$P_t = total\ number\ of\ contracts \times value\ per\ contract = Ƀ\frac{NL}{S_t}$ $entry\ value\ of\ position = P_0 = Ƀ\frac{NL}{S_0}$
Margin Account	Initial Margin Requirement	I	To open a position, margin balance must be at least equal to the initial margin requirement $M_0 \geq I$, $I(Ƀ) = \left(1.00\% \times \frac{N}{S_0}\right) + \left(0.075\% \times \frac{N}{S_0}\right) + \left(0.075\% \times N\frac{L+1}{S_0 L}\right)$
	Margin Account Wallet Balance	(M_t)	By definition of leverage, $initial\ margin\ M_0(Ƀ) = \frac{total\ position\ size}{leverage} = \frac{P_0}{L} = \frac{N}{S_0}$ $Margin\ Balance_t = Wallet\ Balance\ M_t + Unrealised\ PNL_t, t \geq 0$
	Maintenance Margin Requirement	(U_t)	To maintain a position, one must satisfy $M_t + Unrealised\ PNL_t \geq U_t$, $U_t(Ƀ) = \left(0.4\% \times \frac{N}{S_0}\right) + \left(0.075\% \times N\frac{L+1}{S_0 L}\right) + \left(r_t \times N\frac{L+1}{S_0 L}\right)$ as of Aug 2020, the taker fee is 0.0750% and the base initial margin for position sizes < Ƀ 200 is 1.00%
Liquidation	Bankruptcy Price	K	Can be considered as spot price (index price) where "all initial margin is lost". We deduce in Appendix 1 $K(\$) = S_0 \times \frac{L}{L+1} = \frac{S_0 L}{L+1}$
	Liquidation Time (Stopping Time)	τ	The stopping time τ is when our position gets liquidated by BitMEX. It happens when the margin balance goes below the maintenance margin requirement. See the exchange's definition in Appendix 2 $\tau = inf\{t \geq 0: margin\ balance\ (Ƀ) < MMR(Ƀ)\}$ $= \inf\left\{t \geq 0: M_t + NL\left(\frac{1}{S_0} - \frac{1}{S_t}\right) < U_t\right\} = \inf\{t \geq 0: S_t < S_t^L\}$
	Liquidation Price	S_t^L	The index price when the position gets liquidated. The liquidation is constant with respect to time when funding and fees are ignored and variable when they are considered

2 Quantitative Description of BitMEX XBTUSD Contract

2.1 Wealth Process, Cash Flows, and Liquidation Price of a Long Position

Assume at time $t = 0$, we enter a **long position** for NL XBTUSD contracts. With leverage L, we only need to pay the price of N contracts, that is, $\$ N$. We are also required to have in the **margin account** at least enough money that satisfies the **initial margin requirement**, $M_0 \geq I$. By definition of leverage, $M_0 = \text{Ƀ}\frac{N}{S_0}$. The margin account will be used to pay or receive funding amounts at each of the future funding timestamps $t = 1, 2, 3, \ldots$ When the **funding rate** is positive, long pays short (we pay), vice versa. The **margin balance ($\boldsymbol{M_t + Unrealised\ PNL_t}$)** has to satisfy at least the **maintenance margin requirement ($\boldsymbol{U_t}$)** at all times, i.e. $M_t + Unrealised\ PNL_t \geq U_t$; if the margin balance is less than the maintenance margin requirement, then the position gets **liquidated** (we are assuming the position size is $< \text{Ƀ}\ 200$, if it is greater, the exchange would recursively lower risk limits instead of an immediate liquidation).

If we leave the number of contracts unchanged, and do NOT withdraw from or add in money to the margin account, the wealth process and cash flows of the position could be described as following.

2.1.1 Wealth Process & Cash Flows

At time 0, the margin account wallet balance is $M_0 = \text{Ƀ}\frac{N}{S_0}$

the position value is $P_0 = \text{Ƀ}\ \frac{NL}{S_0}$

The initial margin requirement is

$$I(\text{Ƀ}) = \left(1.00\% \ \times \frac{N}{S_0}\right) + \left(0.075\% \times \frac{N}{S_0}\right) + \left(0.075\% \times N\frac{L+1}{S_0 L}\right)$$

Assume that no liquidation occurs up to the n^{th} funding timestamp, $t = n$, then

$$M_n = M_{n-1} - \frac{NL}{S_n}r_n = M_0 - \sum_{i=1}^{n}\frac{NL}{S_i}r_i = \text{Ƀ}\ \frac{N}{S_0} - \sum_{i=1}^{n}\frac{NL}{S_i}r_i$$

$$P_n = \text{Ƀ}\ \frac{NL}{S_n}$$

where $\text{Ƀ}\ \left(-\frac{NL}{S_n}r_n\right)$ is the funding amount received at t_n.

2.1.2 Liquidation Price

Assuming there are no interests on the margin, the time 0 value of the cash flow up to $t = n$ is:

$$\mathbb{E}(V_n) - V_0 = \mathbb{E}(P_n + M_n) - (P_0 + M_0) = \mathbb{E}\left(\frac{NL}{S_n} + \frac{N}{S_0} - \sum_{i=1}^{n} \frac{NL}{S_i} r_i\right) - \frac{NL}{S_0} - \frac{N}{S_0}$$
$$= \mathbb{E}\left(\frac{NL}{S_n} - \sum_{i=1}^{n} \frac{NL}{S_i} r_i\right) - \frac{NL}{S_0} = NL\left(\mathbb{E}\left(\frac{1}{S_n} - \sum_{i=1}^{n} \frac{1}{S_i} r_i\right) - \frac{1}{S_0}\right)$$

The maintenance margin requirement at $t = n$ is:

$$U_n(\text{Ƀ}) = \left(0.4\% \times \frac{N}{S_0}\right) + \left(0.075\% \times N \frac{L+1}{S_0 L}\right) + \left(r_n \times N \frac{L+1}{S_0 L}\right)$$

Liquidation Means the margin balance goes below the maintenance margin requirement

$$Margin\ Balance\ =\ Wallet\ Balance\ (M_t) +\ Unrealised\ PNL < MMR(U_t)$$

the **threshold** is when $Margin\ Balance = MMR$, or equivalently,

$$PnL\ (\text{Ƀ}) =\ MMR(\text{Ƀ}) - Wallet\ Balance\ (\text{Ƀ})$$

$$\Rightarrow NL\left(\frac{1}{S_0} - \frac{1}{S_n^L}\right) = \left(0.4\% \times \frac{N}{S_0}\right) + \left(0.075\% \times N \frac{L+1}{S_0 L}\right) + \left(r_t \times N \frac{L+1}{S_0 L}\right) - \left(\frac{N}{S_0} - \sum_{i=1}^{n} \frac{NL}{S_i} r_i\right)$$

$$\Rightarrow \boldsymbol{S_t^L} = \$\ \frac{\boldsymbol{L^2 S_0}}{\boldsymbol{L^2\left(1 - \sum_{i=1}^{t} \frac{S_0}{S_i} r_i\right) + L(99.525\% - r_t) - 0.075\% - r_t}}$$

2.1.3 Special Case: Without Funding & Fees

At time 0, the margin account wallet balance is

$$M_0 = \text{Ƀ} \frac{N}{S_0}$$

the position value is

$$P_0 = \text{Ƀ}\ \frac{NL}{S_0}$$

The initial margin requirement is

$$I(\text{Ƀ}) = 1.00\%\ \times \frac{N}{S_0}$$

Assume that no liquidation occurs up to the n^{th} funding timestamp, t_n, then

$$M_n = M_0 = \text{Ƀ} \frac{N}{S_0}$$

$$P_n = Ƀ\ \frac{NL}{S_n}$$

$$U_t(Ƀ) = 0.4\% \times \frac{N}{S_0}$$

Liquidation happens at t_n when

$$M_n + NL\left(\frac{1}{S_0} - \frac{1}{S_n}\right) = U_n$$

$$\Rightarrow NL\left(\frac{1}{S_0} - \frac{1}{S^L}\right) = 0.4\% \times \frac{N}{S_0} - \frac{N}{S_0}$$

$$\Rightarrow \boldsymbol{S^L} = \$\ \frac{\boldsymbol{LS_0}}{\boldsymbol{L + 99.6\%}}$$

2.2 Funding Rates & Index Prices Properties

2.2.1 Funding Rates Distribution

We use the data for XBTUSD 8-hourly funding rate history from 2016-06-05 4:00 GMT to 2020-09-02 20:00 GMT (4653 funding timestamps) from BitMEX website (Funding History, 2020).

Figure 3 shows the kernel density graph of funding rate frequency with 500 bins. The distribution resembles a normal distribution with mean: 1.3066×10^{-4}, variance: 1.3549×10^{-6}. The spike occurs at the mode funding rate: 1×10^{-4}.

Figure 4 shows the kernel density graph of funding rate frequency with 500 bins when we exclude all occurrences of funding rate at 1×10^{-4}. The distribution resembles a multimodal distribution with mean: 1.4497×10^{-4}, variance: 1.9868×10^{-6}. The three peaks from left to right are centered at: $-3.77 \times 10^{-3}, 6 \times 10^{-5}, 3.75 \times 10^{-3}$.

Figure 5 shows the fluctuations of 30-day-mean and variance distribution of funding rates. The distribution of mean funding rates shows a mean-reverting character.

Figure 6 shows that funding rate has a low level of autocorrelation for lags greater than 3 funding timestamps. Therefore, future funding rates have low dependency on the past funding rates.

The normal distribution and mean-reverting character of the funding rates indicate that we can simulate funding rate as an Ornstein–Uhlenbeck process described in section 3.2.

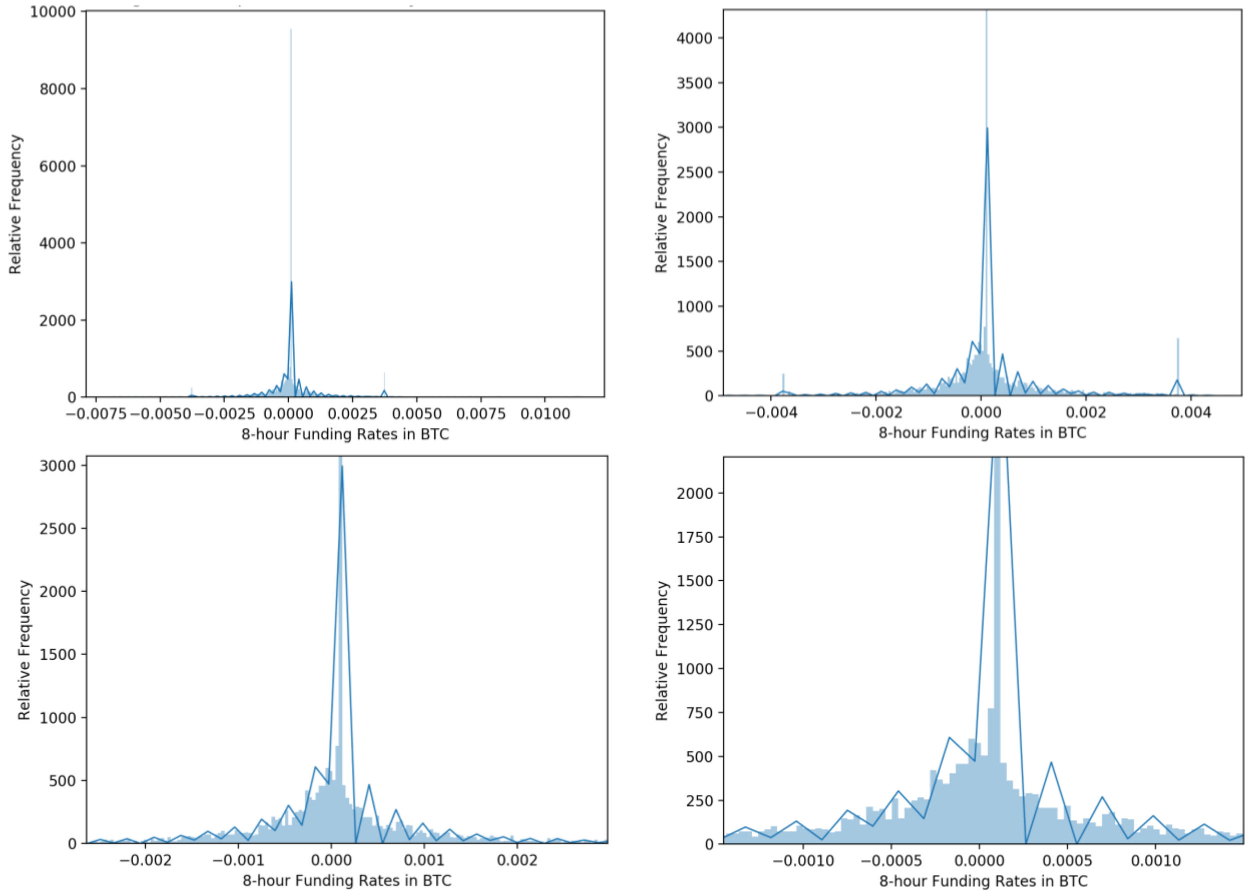


Figure 3. Kernel Density Plot of Funding Rate Frequencies for 1551 Days and 4653 Funding Timestamps

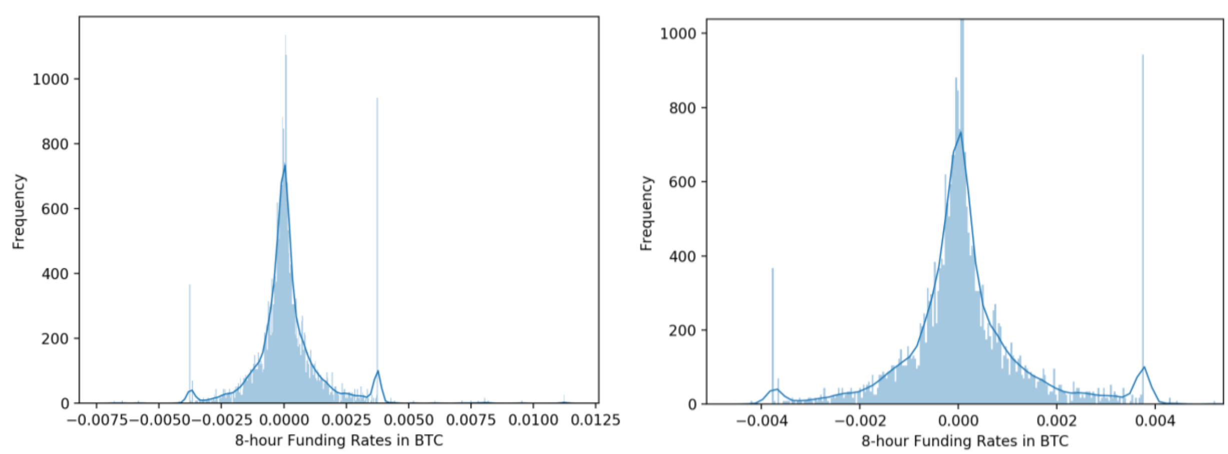
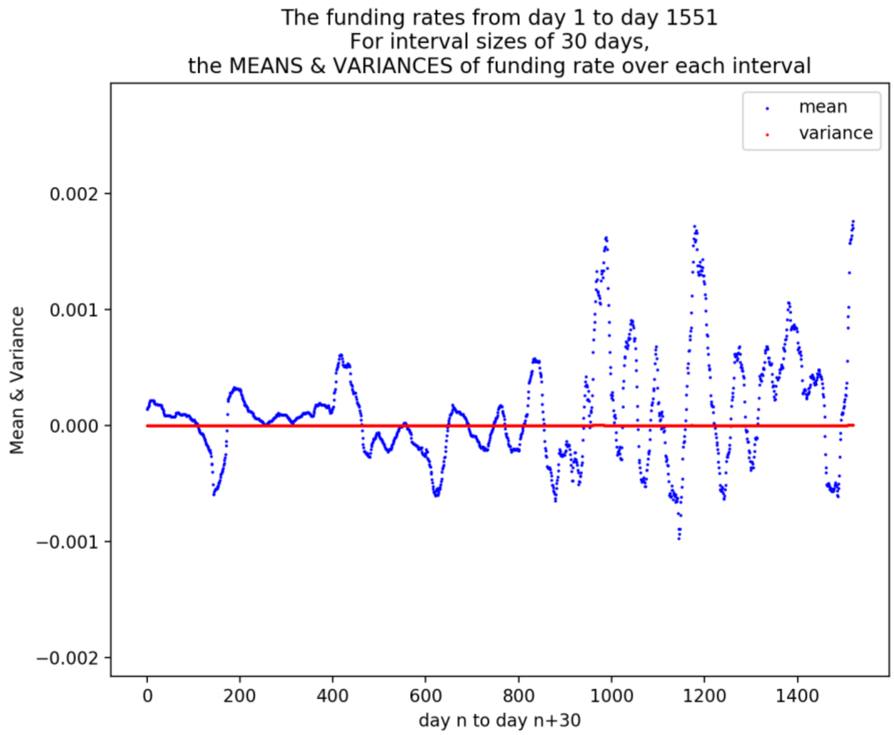
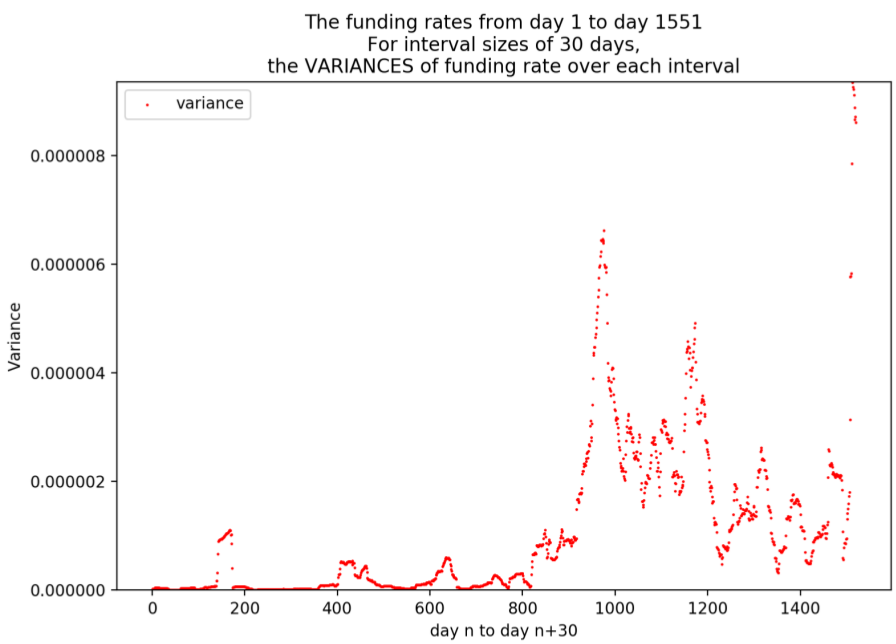


Figure 4. Kernel Density Plot of Funding Rate Frequencies for 1551 Days and 3172 Funding Timestamps (excluding the 1481 observations that funding rates are 0.0001)



a.



b.

Figure 5. a. 30-day-mean and variance distribution of funding rates for 4653 funding timestamps. b. zoomed in the variance distribution in a.

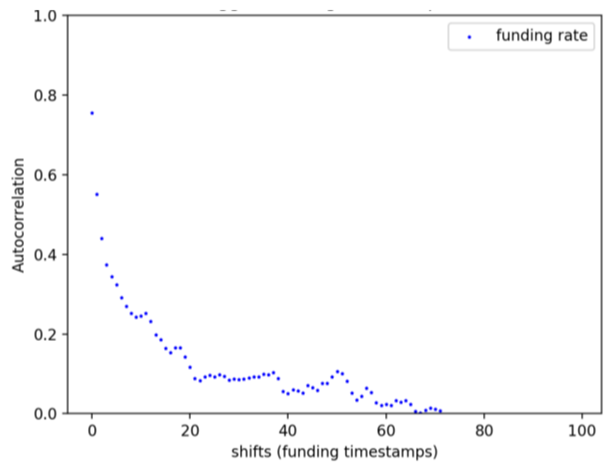


Figure 6. Autocorrelations of Funding Rates For 4653 Funding Timestamps with Lagged Funding Timestamps from 1 to 100

2.2.2 Index Price Distribution

We use data from 2019-01-01 1:01 GMT to 2020-04-18 6:08 GMT (1420 funding timestamps) for XBTUSD 8-hourly index price (these are (high + low) / 2 of index prices at each funding timestamp).

Figure 7 shows that future index prices are highly correlated with past index prices. Intuitively, the exchange rate of Bitcoin and USD follows a continuous path and do not fluctuate haphazardly over every 8 hours.

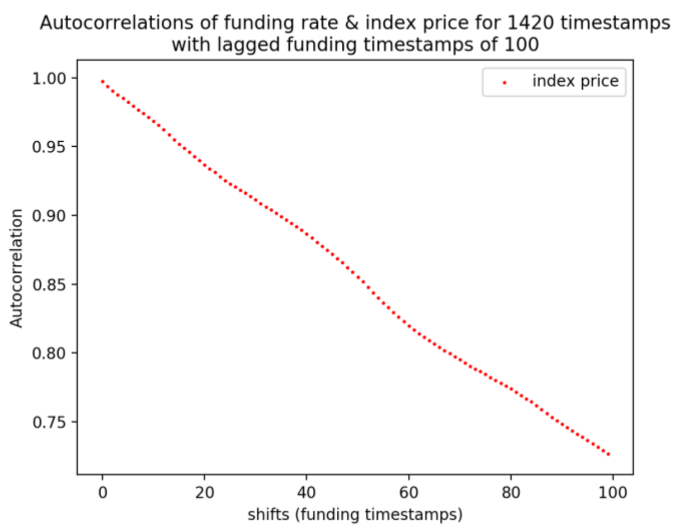


Figure 7. Autocorrelations of index prices for 1420 funding timestamps with lagged funding timestamps of 1 to 100

2.2.3 Funding Rates vs Index Price Regression

Model & Variables

We regress funding rates on index price $x_{1,t}$, index price squared $x_{2,t}$, and lagged index price $x_{4,t-1}, x_{5,t-2}, x_{6,t-3}$ .... lagged index price $x_{i+3,t-i}$ is the index price i funding timestamps before t.

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \sum_{k=4}^{3+lagTime} \beta_k x_{k,(t-k+3)} + \epsilon_t, \quad t = lagTime + 1, \dots, 1420$$

where $lagTime$ is the number of past funding timestamps we want to consider in the regression.

Table 2. Regression Variables

Symbol	Variable	Descriptions
Y_t	Funding Rate	The 8-hourly funding rate at funding timestamp t
$x_{1,t}$	Index Price	The index price at funding timestamp t multiplied by 10^{-6}
$x_{2,t}$	Index Price Squared	The square of the index price at funding timestamp t multiplied by 10^{-12}. $x_{2,t} = x_{1,t}^2$
$x_{3,t}$	Dow Jones Industrial Average	The daily Dow Jones Industrial Average at the day (or the last trading day) of the timestamp t multiplied by 10^{-4}.
$x_{i+3,t-i}$	Lagged Time Index Price	The index price i funding timestamps before t multiplied by 10^{-4}. $i \in [0, \infty)$

We run the regression on the historical data from 2019-01-01 1:01 GMT to 2020-04-18 6:08 GMT (1420 funding timestamps) XBTUSD 8-hourly funding timestamps (complete R regression report in Appendix 3) and daily Dow Jones Industrial Average over the same period of time (Dow Jones Industrial Average, 2020). We choose Dow Jones Industrial Average as one of the variables because it indicates the economy and financial market performance of US industry. Although the decentralized feature of blockchain makes it difficult to obtain the geographic distribution of Bitcoin traders, the fact that 66.86% of global Bitcoin ATMs are located in the United States (Topic: Bitcoin, 2020) indicates Bitcoin's popularity among American traders. Furthermore, the global economy and internet technology are dominated by developed countries like the United States, giving Dow Jones Industrial Average more credibility as a potential variable for Bitcoin spot prices.

We tested the goodness of fit by a control study in **Table 4**. The models are in **Table 3.**

Table 3. Control Variable Regression Models

Model #	Control	Formula
1	lagTime = 0	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t$
2	lagTime = 0 & without Dow Jones Term	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \epsilon_t$
3	lagTime = 0 & without Index Price Term	$Y_t = \beta_0 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t$
4	lagTime = 0 & without Index Price Squared Term	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_3 x_{3,t} + \epsilon_t$
5	lagTime = 1	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t-1} + \epsilon_t$
6	lagTime = 2	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t-1} + \beta_5 x_{5,t-2} + \epsilon_t$
7	lagTime = 3	$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t-1} + \beta_5 x_{5,t-2} + \beta_6 x_{6,t-3} + \epsilon_t$

Since the index price has a high value of autocorrelation (section 2.2.2), we expect there to be not much improvement in the accurateness of regression by having n $lagTime$ variables to having none. Compare Model 1, 5, 6, 7 for $lagTime = 0,1,2,3$ respectively. **Figure 4** shows that the adjusted R-square of $lagTime = 0$ (0.2709) is very close to that of $lagTime = 1$ (0.2714), and the residual standard error of $lagTime = 0$ is 0.0003987 is equal to that of $lagTime = 1$ (Model 1, 5); this can be explained by the close to perfect autocorrelation between the current index price and the index price at the last funding timestamp. Although both the adjusted R-square and the residual standard error has slight improvements as $lagTime$ variables increase from 1 to 3 (Model 5, 6, 7), this is not robust to indicate an appreciable improvement in the goodness of fit.

A comparison between Model 1 & Model 2 shows that including Dow Jones term would greatly increase the model's goodness to fit; a comparison between Model 1 & Model 3, Model 1 & Model 4 show that both index price term and index price squared term could improve the model's fitness, and that the improvement is more apparent with incorporating the index price squared term. Comparisons between Model 1 & Model 2, 3, 4 show that incorporating the Dow Jones term decreases the residual standard error the most. Comparisons between Model 1 & Model 5, 6, 7 show that including lagTime terms are not necessary for the goodness of fit. This conclusion is also supported by the high autocorrelation level between current and past index prices.

Please refer to Appendix 3 for complete R reports.

Table 4. Control Variable Regression Models Test Results

Criterion	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
Adjusted R^2	0.2709	0.2087	0.2564	0.2398	0.2714	0.285	0.3015
Residual standard error	0.0003987 DF = 1416	0.0004154 DF = 1417	0.0004027 DF = 1417	0.0004071 DF = 1417	0.0003987 DF = 1414	0.0003951 DF = 1412	0.0003907 DF = 1410
p-value	The p-values are all $< 2.2 \times 10^{-16}$						

Regression Results

According to the control test, we choose Model 1:

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t$$

Table 5 shows the quantile distribution of regression variables. The minimum of index price is 3358, occurred on 29th January 2019, while the maximum of index price is 13481, occurred on 26th June 2019 (Bitcoin Price Index Monthly 2012-2020, 2020). As we have explained earlier, BitMEX's .BXBT index price is a weighted average of Bitcoin / USD exchange rates over five major exchanges. Given the volatility of Bitcoin spot markets, it is not uncommon for the exchange rate to fluctuate by $10000 / Ƀ over 16 months.

Table 5. Quantile Distribution of Variables Times $\times 10^{-4}$

Variable	Min	1Q	Median	3Q	Max	Std Dev
Y_t	-37.5	-0.58	1	1	31.59	4.6679
$x_{1,t}$	33.5784	52.9129	79.2903	93.9194	134.8101	24.3789
$x_{2,t}$	0.1128	0.2800	0.6287	0.8821	1.8174	0.3583
$x_{3,t}$	1859.1930	2561.5615	2643.8480	2726.9971	2955.1420	187.7828
ϵ_t	-0.0036	-0.0002	0.0000	0.0002	0.0029	-

The regression result for $lagTime = 0$:

$$Y_t = -0.0015614 - 0.1460681\, x_{1,t} + 14.2736774\, x_{2,t} + 68.6675722\, x_{3,t}$$

Table 6 summarizes the regression results. The p-value (p-value: $< 2.2 \times 10^{-16}$) is much smaller than 0.01, meaning there is significant linear relationship between the variables in the regression. The regression coefficients for $x_{1,t}$ and $x_{2,t}$ are both significant under <0.1% significance level.

Consider that for standard vanilla futures contracts, the long position pays the short position daily the amount of daily increases in the underlying asset price. For XBTUSD contract, the underlying asset is the index price. The primary function of funding is to tether the Bitcoin / USD spot rate, which is reflected through the index price on BitMEX. The regression coefficient of index price squared ($\beta_2 = 14.2736774$) is much greater than that of index price ($\beta_1 = -0.146068$), thus the model indicates that when the index price increases, funding rate tends to increase, matching the exchange's rule that the long position needs to pay the short position when funding rate is positive.

Table 6. Regression coefficients

Variable	**Coefficient**
$intercept$	-0.0015614 ***
	(-9.359)
$x_{1,t}$	-0.1460681 ***
	(-5.411)
$x_{2,t}$	14.2736774 ***
	(7.849)
$x_{3,t}$	68.6675722 ***
	(11.044)
Sample	1420
Adjusted R^2	0.2709

Note: t values are inside the parenthesis; *, **, *** means <5%, <1%, <0.1% significance level

Pearson correlation coefficient

Table 7 measures the strength of a linear association between the variables. Our regression model is reliable given that all of the Pearson coefficients for different variable pairs are below 0.5 except for between $x_{1,t}$ and $x_{2,t}$. This is because by definition, $x_{2,t} = x_{1,t}^2$. We decide to keep them in the regression model because by control studies we have shown that keeping both these variables would improve appreciable goodness of fit.

Table 7. Pearson Correlation Coefficient of Variables

Variable	Y_t	$x_{1,t}$	$x_{2,t}$	$x_{3,t}$
Y_t	1			
$x_{1,t}$	0.4318	1		
$x_{2,t}$	0.4510	0.9866	1	
$x_{3,t}$	0.3903	0.4120	0.3910	1

3 Liquidation in Relation to Leverage

3.1 Set up

Consider we enter a long position on 1 contract at time $t = 0$, with leverage $L \in [1,100]$, and initial index price S_0. We do NOT adjust our position and do NOT take out or add in money to the margin account after time 0. Our position gets liquidated as soon as the index price reaches or goes below S_t^L($). Define stopping time

$$\tau = \inf\{t \geq 0 : S_t < S_t^L\}$$

we want to evaluate $\mathbb{E}(\tau)$ and $\mathbb{P}(\tau \leq T)$, which are the expected time for the position to be liquidated and the probability of getting liquidated before time $T > 0$.

3.2 Assumptions

Assume under risk-neutral probability $\mathbb{Q}$,

- The **index price** ($\boldsymbol{S_t}$) follows the Geometric Brownian Motion:

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

$$\Leftrightarrow S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t}, t \geq 0$$

  note that often times, σ_t is not a constant, but a mean-reverting process. Here we simplify the situation by taking σ as a constant

  Path for simulation:

$$S_{t+dt} = S_t e^{\sigma \epsilon_{t+dt} + \left(\mu - \frac{\sigma^2}{2}\right)dt}, \quad \epsilon_{t+dt} = W_{t+dt} - W_t \sim N\left(0, \sqrt{dt}\right)$$

- The **funding rate** ($\boldsymbol{r_t}$) is Gaussian and follows the Ornstein–Uhlenbeck process Vasicek model

$$dr_t = a(b - r_t)dt + \xi dW_t^{(1)}$$

$$\Leftrightarrow\ r_t = r_0 e^{-at} + b(1 - e^{-at}) + \xi \int_0^t e^{-a(t-s)} dW_s^{(1)}$$

where b is the long-term mean level, a is speed of reversion, ξ is instantaneous volatility, $\frac{\xi^2}{2a}$ is long-term variance, $\left(W_t^{(1)}\right)$ is a Weiner process under $\mathbb{Q}$.
Path for simulation:

$$r_{t+du} = (1-\rho)\zeta + \rho r_t + (1-\rho^2)^{0.5}\sigma_r \varepsilon_{t+du}, \quad \varepsilon_{t+du} = W_{t+du} - W_t \sim N\left(0, \sqrt{du}\right)$$

Note: In the previous section on Funding Rate Properties, we've shown that historical data for the funding rates resembled normal distribution. Therefore, in theory, we could model funding rates with respect to time as a mean-reverting process.

3.3 Without Funding and Fees

3.3.1 Expected Liquidation Time $\mathbb{E}(\tau)$

Formula 1 (Lower Boundary Only)

The expected liquidation time of long position on NL XBTUSD contracts when there is no funding or fees is

$$\mathbb{E}(\tau) = BS_0^{1-\frac{2\mu}{\sigma^2}}\left(1 - \left(\frac{L}{L+0.996}\right)^{1-\frac{2\mu}{\sigma^2}}\right) + \frac{2\log\frac{L+99.6\%}{L}}{\sigma^2 - 2\mu}, \sigma > 0, L \in [1,100]$$

where $\tau = \inf\{t \geq 0 : S_t < S_t^L\}, S^L = \frac{LS_0}{L+99.6\%}$, *for some constant* $B \in \mathbb{R}$, $\mu < 0$ *for* $\mathbb{E}(\tau)$ *to be finite.*

Procedures

Recall the constant liquidation price $S^L = \$\ \frac{LS_0}{L+99.6\%}$

$$\tau = \inf\{t \geq 0 : S_t < S^L\} \approx \inf\left\{t \geq 0 : \$\ S_t \leq \$\ \frac{LS_0}{L+99.6\%}\right\}$$

<u>*Martingale*</u>

Define martingale on $[0, \tau]$,

$$Y_t = \mathbb{E}_t(\tau) = \mathbb{E}_t(\tau - t) + t = f(S_t) + t, t \leq \tau$$

Satisfies $f(S^L) = 0$ since $Y_\tau = \mathbb{E}_\tau(\tau) = 0 + \tau$

Verification: $f(S_0) = Y_0 = \mathbb{E}(Y_\tau) = \mathbb{E}(f(S_\tau) + \tau) = \mathbb{E}(\tau)$

Since $dS_t = S_t(\mu dt + \sigma dW_t)$, so $(dS_t)^2 = S_t^2\sigma^2 dt$

by Itô's formula,

$$dY_t = 1dt + f_s(S_t)dS_t + \frac{1}{2}f_{ss}(S_t)(dS_t)^2 = 1dt + f_s(S_t)S_t(\mu dt + \sigma dW_t) + \frac{1}{2}f_{ss}(S_t)S_t^2\sigma^2 dt$$

Thus,

$$Y_t = Y_0 + \int_0^t \left(\frac{1}{2}f_{ss}(S_t)S_t^2\sigma^2 + f_s(S_t)S_t\mu + 1\right)dt + \int_0^t (f_s(S_t)S_t\sigma)\, dW_t$$

Since Y_t is a martingale, the drift term (time-dependent term) is 0, thus we have the following ODE:

$$\begin{cases} \frac{\sigma^2}{2}x^2 f_{xx}(x) + \mu x f_x(x) + 1 = 0, & x \in [S^L, \infty) \\ f(S^L) = 0, & or\ preisely,\ \lim_{x^- \to M} f(x) = 0 \end{cases}$$

<u>Solve ODE</u>

Homogeneous solution: $f_{homo}(x) = A + Bx^{1-\frac{2\mu}{\sigma^2}}$ for some constant A, B

Particular solution: $f_{particular}(x) = \frac{2\ ln(x)}{\sigma^2 - 2\mu}$

So, $f(x) = A + Bx^{1-\frac{2\mu}{\sigma^2}} + \frac{2\ ln(x)}{\sigma^2 - 2\mu}$

Plugging in boundary condition $f(S^L) = 0$ we get $A = -B(S^L)^{1-\frac{2\mu}{\sigma^2}} - \frac{2\ ln(S^L)}{\sigma^2 - 2\mu}$

Plugging in $S^L = \frac{LS_0}{L+99.6\%}$ we get **Formula 1**

<u>Analysis</u>

Consider the scenario when $B = 0$, set $g(L,\sigma) = \frac{2\ \log\frac{L+99.6\%}{L}}{\sigma^2 - 2\mu}$,

then $\mathbb{E}(\tau) = f_{particular} = \frac{2\ \log\frac{L+99.6\%}{L}}{\sigma^2 - 2\mu} = g(L,\sigma)$.

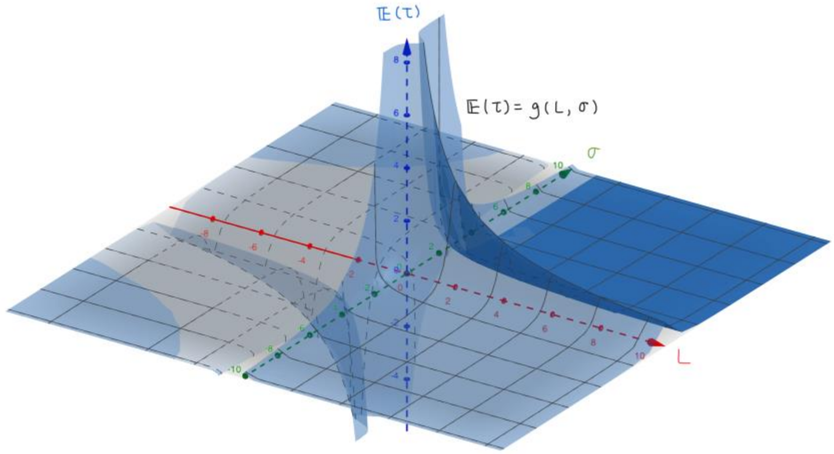
For $g(L,\sigma) = \frac{2\ \log\frac{L+99.6\%}{L}}{\sigma^2 - 2\mu}$ to be positive, we must have $\mu < \frac{\sigma^2}{2}$.

Given that S_t only has a lower bound at S^L, for positive drift $\mu > 0$, we have $\mathbb{P}(\tau = \infty) > 0$, thus the value of $\mathbb{E}(\tau)$ is unbounded. So, for $\mu < 0$, $\mathbb{P}(\tau < \infty) = 1$, and $\mathbb{E}(\tau)$ is bounded.

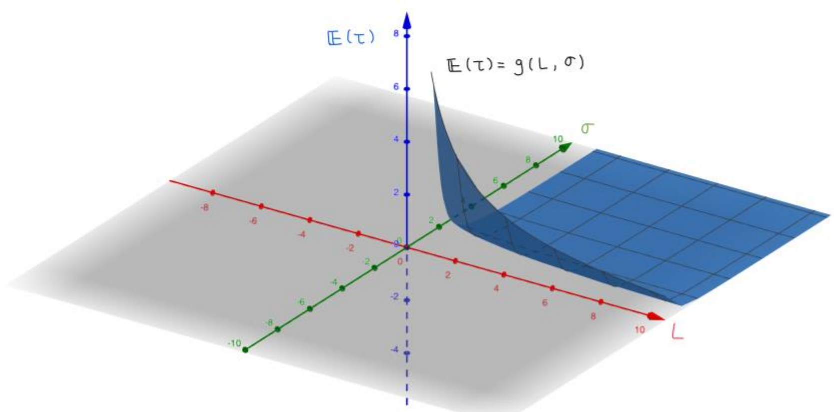
Observation: when $\sigma > 0$, $\sigma \to 0$, and $\mu < 0$, we have $S_0^{1-\frac{2\mu}{\sigma^2}}\left(1 - \left(\frac{L}{L+0.996}\right)^{1-\frac{2\mu}{\sigma^2}}\right) \to \infty$, $0 < \frac{2\ \log\frac{L+99.6\%}{L}}{\sigma^2 - 2\mu} \ll \infty$, then we must have $B \to 0$ for $\mathbb{E}(\tau)$ to be bounded.

Figure 8 are the plots of $g(L,\sigma)$ (x-axis: leverage L, y-axis: σ, z-axis: $\mathbb{E}(\tau)$, $\mu = -0.1$). Since $\sigma^2 > 0$ in $g(L,\sigma)$, the plot is symmetric along the x-axis (leverage L). We only consider plot b. where $L \in [1,100], \sigma > 0$. Notice that the effect of L and σ on $\mathbb{E}(\tau)$ is most apparent for small L and σ, for which $\mathbb{E}(\tau)$ decreases rather quickly with increments of L or σ. For larger L or σ, $\mathbb{E}(\tau)$ decreases much slower with increments of L or σ. Mathematically, this can be explained by

$\frac{\partial^2 g(L,\sigma)}{\partial L^2}, \frac{\partial^2 g(L,\sigma)}{\partial \sigma^2} > 0$. Financially, this indicates that the risk of liquidation increases most significantly for lower leverage or lower volatility ranges, and remains rather staple for higher leverage or higher volatility ranges. Therefore, when the volatility is low, investors at lower leverage ranges would be more cautious about increasing their leverage, which would sharply decrease their expected liquidation time.



a.



b.

Figure 8. Plot of the Particular Solution (x-axis: leverage L, y-axis: σ, z-axis: $\mathbb{E}(\tau)$, $\mu = -0.1$),
a. L, σ are unconstrained; b. $L \in [1,100], \sigma > 0$

Formula 2 (Upper & Lower Boundaries)

If we impose an upper limit S^U, (sell our position as soon as the index price reaches S^U), in addition to the forced liquidation at $S^L = \frac{LS_0}{L+99.6\%}$, the expected exit time $\tau_2 = \inf\{t \geq 0: S_t < S^L \text{ or } S_t > S^U\}$ when there is no funding or fees is

$$\mathbb{E}(\tau_2) = \frac{2\left(ln\left(\frac{LS_0}{S^U(L+0.996)}\right)\right)S_0^{1-\frac{2\mu}{\sigma^2}}}{(\sigma^2-2\mu)\left((S^U)^{1-\frac{2\mu}{\sigma^2}} - \left(\frac{LS_0}{L+0.996}\right)^{1-\frac{2\mu}{\sigma^2}}\right)}\left(1-\left(\frac{L}{L+0.996}\right)^{1-\frac{2\mu}{\sigma^2}}\right) + \frac{2\,\log\frac{L+0.996}{L}}{\sigma^2-2\mu}$$

where $\sigma > 0, L \in [1,100]$

Procedures

Consider an upper boundary S^U such that we exit the position at

$$\tau_2 = \inf\{t \geq 0: S_t < S^L \text{ or } S_t > S^U\} \approx \inf\left\{t \geq 0: \$\, S_t \leq \$\, \frac{LS_0}{L+99.6\%}, \$\, S_t > \$\, S^U\right\}$$

Then $f(S^U) = 0$.

Plugging in the equation from the previous section,

$$f(S^U) = -B(S^L)^{1-\frac{2\mu}{\sigma^2}} - \frac{2\,ln(S^L)}{\sigma^2-2\mu} + B(S^U)^{1-\frac{2\mu}{\sigma^2}} + \frac{2\,ln(S^U)}{\sigma^2-2\mu} = 0$$

We get

$$B = \frac{\frac{2\,ln(S^L)}{\sigma^2-2\mu} - \frac{2\,ln(S^U)}{\sigma^2-2\mu}}{(S^U)^{1-\frac{2\mu}{\sigma^2}} - (S^L)^{1-\frac{2\mu}{\sigma^2}}} = \frac{2\left(ln(S^L) - ln(S^U)\right)}{(\sigma^2-2\mu)\left((S^U)^{1-\frac{2\mu}{\sigma^2}} - (S^L)^{1-\frac{2\mu}{\sigma^2}}\right)}$$

Plug in **Formula 1**, we get **Formula 2**

Note: taking limit $S^U \to \infty$ in **Formula 2** will get **Formula 1**

3.3.2 Probability of Liquidation Before T: $\mathbb{P}(\tau \leq T)$

Formula 3

The probability of getting liquidated before time T is

$$\mathbb{P}(\tau \leq T) \approx 1 - N\left(\frac{\ln\left(\frac{L+99.6\%}{L}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}\right) + \exp\left(2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)\ln\left(\frac{L}{L+99.6\%}\right)\right) N\left(\frac{\ln\left(\frac{L}{L+99.6\%}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T}\right)$$

where $T > 0, \sigma > 0, L \in [1,100]$.

Procedures

Define martingale on $[0, \tau]$,

$$Y_t = \mathbb{P}_t(\tau \leq T) = f(T - t, S_t)$$

Satisfies $f(0, x) = 0$, $f(t, S^L) = 1$, and $\lim_{S \to \infty} f(t, S) = 0$

By Itô's formula,

$$\begin{aligned} dY_t &= -f_t(T-t, S_t)dt + f_s(T-t, S_t)dS_t + \frac{1}{2} f_{ss}(T-t, S_t)(dS_t)^2 \\ &= -dt + f_s(T-t, S_t)S_t(\mu dt + \sigma dW_t) + \frac{1}{2} f_{ss}(S_t)S_t^2\sigma^2 dt \end{aligned}$$

Thus,

$$Y_t = Y_0 + \int_0^t \left(\frac{1}{2} f_{ss}(T-t, S_t)S_t^2\sigma^2 + f_s(T-t, S_t)S_t\mu - f_t(T-t, S_t) \right) dt + \int_0^t \left(f_s(T-t, S_t)S_t\sigma \right) dW_t$$

Since Y_t is a martingale, the drift term (time-dependent term) is 0, thus we have the following ODE:

$$\begin{cases} \dfrac{\sigma^2}{2} x^2 f_{xx}(t, x) + \mu x f_x(t, x) - f_t(t, x) = 0, & (t, x) \in (0, T) \times (S^L, \infty) \\ f(0, x) = 0, & x \in (S^L, \infty) \\ f(t, S^L) = 1, & t \in [0, T] \\ \lim_{x \to \infty} f(t, x) = 0, & t \in [0, T] \end{cases}$$

The solution to the set of partial differential equations is given in an article on first hitting time. Guillaume (2014) has shown that for a geometric BM with $S_t = S_0 e^{\sigma_t dW_t + \left(\mu_t - \frac{\sigma_t^2}{2}\right)dt}$, the probability that X_t will not hit boundary $M, M < x_0$ during finite time interval $[0, T]$ and that it will be above $k, k > M$ at time T is given by:

$$\mathbb{P}\left(\inf_{0 \leq t \leq T} X_t > M, X(T) > k\right)$$
$$\approx N\left(\frac{\ln\left(\frac{S_0}{k}\right)}{\sqrt{\int_0^T \sigma_t^2 dt}} + \sqrt{\int_0^T \frac{\left(\mu_t - \frac{\sigma_t^2}{2}\right)^2}{\sigma_t^2} dt} \right) - \exp\left(\frac{2\ln\left(\frac{M}{S_0}\right)\sqrt{\int_0^T \frac{\left(\mu_t - \frac{\sigma_t^2}{2}\right)^2}{\sigma_t^2} dt}}{\sqrt{\int_0^T \sigma_t^2 dt}} \right) N\left(\frac{\ln\left(\frac{M^2}{kS_0}\right)}{\sqrt{\int_0^T \sigma_t^2 dt}} + \sqrt{\int_0^T \frac{\left(\mu_t - \frac{\sigma_t^2}{2}\right)^2}{\sigma_t^2} dt} \right)$$

where $N(.)$ is the CDF of $N(0,1)$ distribution.

We consider constant μ and σ,

$$\mathbb{P}\left(\inf_{0\le t\le T} X_t > M, X(T) > k\right)$$
$$\approx N\left(\frac{\ln\left(\frac{S_0}{k}\right)}{\sqrt{\int_0^T \sigma^2 dt}} + \sqrt{\int_0^T \frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{\sigma^2} dt}\right) - \exp\left(\frac{2\ln\left(\frac{M}{S_0}\right)\sqrt{\int_0^T \frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{\sigma^2} dt}}{\sqrt{\int_0^T \sigma^2 dt}}\right) N\left(\frac{\ln\left(\frac{M^2}{kS_0}\right)}{\sqrt{\int_0^T \sigma^2 dt}} + \sqrt{\int_0^T \frac{\left(\mu-\frac{\sigma^2}{2}\right)^2}{\sigma^2} dt}\right)$$
$$= N\left(\frac{\ln\left(\frac{S_0}{k}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right) - \exp\left(2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\ln\left(\frac{M}{S_0}\right)\right) N\left(\frac{\ln\left(\frac{M^2}{kS_0}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right)$$

and take $k = M + \varepsilon$, for $\varepsilon \to 0$

$$\mathbb{P}(\tau > T) = \mathbb{P}\left(\inf_{0\le t\le T} X_t > M, X(T) > M\right) = \lim_{\varepsilon\to 0} \mathbb{P}\left(\inf_{0\le t\le T} X_t > M, X(T) > M+\varepsilon\right)$$
$$\approx N\left(\frac{\ln\left(\frac{S_0}{M}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right) - \exp\left(2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\ln\left(\frac{M}{S_0}\right)\right) N\left(\frac{\ln\left(\frac{M}{S_0}\right)}{\sigma\sqrt{T}} + \left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}\right)$$
$$\mathbb{P}(\tau \le T) = 1 - \mathbb{P}(\tau > T)$$

Plug in $M = \frac{S_0 L}{L+99.6\%}$, we get **Formula 3**

3.4 Account for Funding and Fees - Simulations

3.4.1 Set Up

Since most investors would anticipate to sell off their positions for cash at some future dates (an "artificial maturity", T) instead of keeping their positions on the exchange indefinitely, it is helpful to know before that "maturity" $T > 0$, the expected liquidation time if liquidation happens $\mathbb{E}(\tau')$ and the probability of getting liquidated $\mathbb{P}(\tau' \le T)$. Therefore, we simulate 2000 paths of (S_t) and (r_t) for every combination of parameters using the simulation formulas as described in **Table 8** (Assumptions) to evaluate the characteristics of liquidation (stopping) time before $T > 0$ for different leverages, $\tau' = \{0 \le t \le T | S_t < S_t^L\}$. **Table 9** gives the constants we will use for all simulations.

Table 8. Simulation Paths Parameters, Variables, and Model

Path	Parameters	Variables	Simulation Model
$\boldsymbol{S_t}$	σ, μ, dt	ϵ_t, S_{t-dt}	$S_{t+dt} = S_t e^{\sigma\epsilon_{t+dt}+\left(\mu-\frac{\sigma^2}{2}\right)dt}$ $\epsilon_{t+dt} = W_{t+dt} - W_t \sim N(0, \sqrt{dt})$
$\boldsymbol{r_t}$	$\sigma_r, \rho, \zeta, du$	ε_t, r_{t-1}	$r_{t+du} = (1-\rho)\zeta + \rho r_t + (1-\rho^2)^{0.5}\sigma_r\varepsilon_{t+du}$ $\varepsilon_{t+du} = W_{t+du} - W_t \sim N(0, \sqrt{du})$

Table 9. Simulation Constants

Constant	Symbol	Value	Description
Maturity	T	80	Total number of time periods
Sampling Time Period	dt	0.125	The rate of sampling in terms of time periods. The total number of samples is $\frac{T}{dt}$
Funding Time Period	du	1	The rate of funding timestamp occurrences in terms of time periods. The total number of funding timestamps is $\frac{T}{du}$. Every funding timestamp is 8 hours, so maturity is $\frac{8T}{du}$ hours
Number of Paths	$numPaths$	2000	The number of paths to simulate for (S_t) and (r_t)
Margin Number of Contracts	N	1	The number of contracts bought initially with our own money. The total number of contracts is $N \times leverage$
Initial Index Price	S_0	10000	The initial index price we choose is close to the average price of a bitcoin
Initial Funding Rate	r_0	0.0001	The initial funding rate we choose is close to the mean of historical funding rates
Funding Rate Cap	cap	0.0045	The absolute funding rate is capped at $75\% \times (Initial\ Margin - Maintenance\ Margin)$. We use an approximated $75\% \times (1\% - 0.4\%) = 0.45\%$

The Python codes for simulations are in Appendix 4. You can try different values for the constants and parameters.

For the reference of parameter values for S_t, we have run a simulation on the data from 2019-01-01 1:01 GMT to 2020-04-18 6:08 GMT (1420 funding timestamps) of XBTUSD 8-hourly funding timestamps. For r_t, we run a simulation on the data from 2016-06-05 4:00 GMT to 2020-09-02 20:00 GMT (4653 funding timestamps). Therefore, the parameters we get could be referenced for 8-hourly periods. The regression formulas we use are consistent with the simulated paths for (S_t) and (r_t) as specified in Assumptions. The regression coefficients are our references for simulation parameters. Since there are random terms drawn from Gaussian distribution, we yield slightly different values for parameters every time; **Table 10** shows the **approximated** average or range of the regression coefficients from running the regressions ten times. Values of σ, $(1-\rho)\zeta$, and ρ, are nearly always concentrated around a certain value, while values for μ and $(1-\rho^2)^{0.5}\sigma$ have more variances every time given the randomness of ϵ_t and ε_t in their terms.

Table 10. Regression for Parameter Reference

Path	Regression Model	Parameters
$\boldsymbol{S_t}$	$y_t = \beta_0 + \beta_1 x_{1t} + error_t$	$\sigma \approx 0.0005$
	based on	$\mu \in [-0.0005, 0.001]$
	$\ln(S_t) - \ln(S_{t-1}) = \sigma\epsilon_t + \left(\mu - \frac{\sigma^2}{2}\right)$	
$\boldsymbol{r_t}$	$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + error_t$	$(1-\rho)\zeta \approx 0.00003$
	based on	$\rho \approx 0.75$
	$r_t = (1-\rho)\zeta + \rho r_{t-1} + (1-\rho^2)^{0.5}\sigma\varepsilon_t$	$(1-\rho^2)^{0.5}\sigma \in [-0.0001, -10 \times 10^{-6}]$

We control the funding rate process under the same set of parameters to understand how leverage effects liquidation under different drift μ and volatility σ of index price S_t.

The model we use for the funding rate path is:

$$r_t = 0.00003 + 0.75\, r_{t-1} - 0.0003\varepsilon_t$$

Figure 9 shows an arbitrary path following the above model. The path resembles the historical funding rates. For non-funding timestamps, the funding rate is considered to be 0.

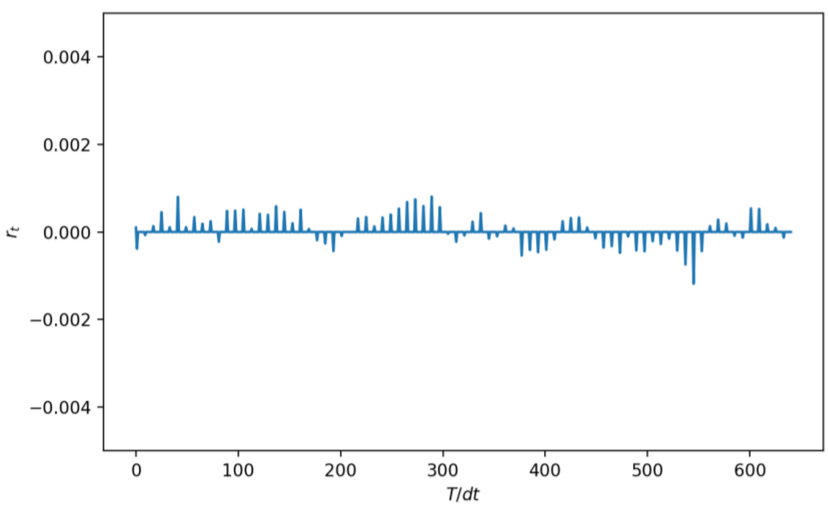


Figure 9. An Arbitrary Simulated Path for r_t

Figure 10 shows ten arbitrary paths following the S_t model with $\sigma = 0.0005$ and $\mu = 0$. The path resembles the historical index price.

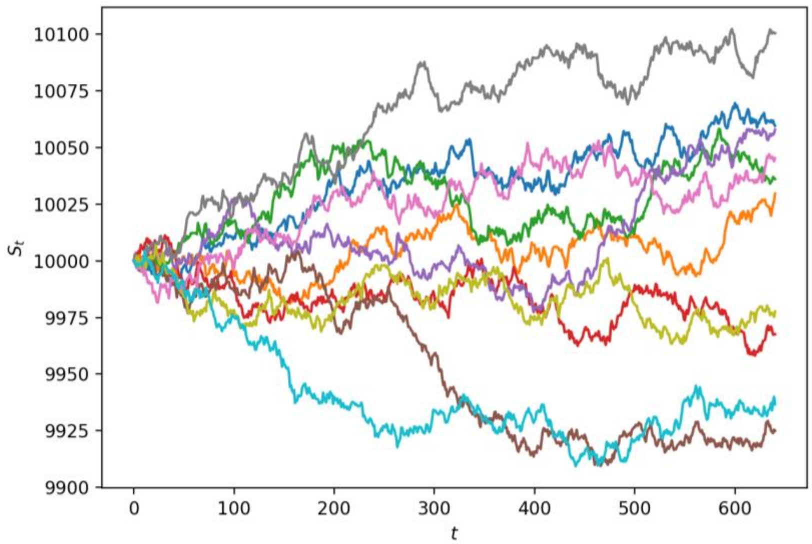


Figure 10. Ten Arbitrary Simulated Paths for S_t with $\sigma = 0.0005$ and $\mu = 0$

Using different combinations of the parameters, drift $\mu \in [-0.5, 0.0005]$ and volatility $\sigma \in [0, 0.5]$ (samples are relatively evenly spaced within these intervals), for index price S_t, we ran 2000 simulations for each leverage in the set $\{5k|k \in [1,20]\}$ for each combination. We obtained 439 valid data points out of a total of 620; these data have parameters in the range $\mu \in [-0.0005, 0.0005]$ and volatility $\sigma \in [0, 0.1]$. The data points we excluded were the combinations that none of the 2000 paths was liquidated before T, thus are considered as outliers and could cause errors in the regression models. The next two sections summarize regression results. The variables selected in the regression models in these sections are based on a control study for a best fit according to adjusted R^2 and is omitted here.

3.4.2 Expected Liquidation Time for Liquidations Before Time T: $\mathbb{E}(\tau')$

The regression model:

$$\mathbb{E}(\tau')_i = \beta_0 + \beta_1\sigma_i + \beta_2\sigma_i^{0.5} + \beta_3\sigma_i^2 + \beta_4\mu_i + \beta_5\mu_i^2 + \beta_6 L_i + \epsilon_i, \quad i = 1, \dots, 440$$

The average for all liquidation before the fixed maturity $T = 80$ follows:

$$\mathbb{E}(\tau')_i = 72.18 + 1587\sigma_i - 551.8\sigma_i^{0.5} - 4006\sigma_i^2 + 4811\mu_i - 1.298 \times 10^7\mu_i^2 - 0.2208L_i + \epsilon_i$$
$$i = 1, \dots, 440$$

Table 11 summarizes the regression results; see Appendix 5 for the R report. The p-value (p-value: $< 2.2 \times 10^{-16}$) is much smaller than 0.01, thus there is significant linear relationship between the variables in the regression.

The coefficient for L is negative, meaning that expected liquidation time decreases as leverage increases.

Recall for drift $\mu < 0$, S_t tends to move downward, thus the probability of getting liquidated in finite time is 1; for $\mu < 0$, positive first-order partial derivative with respect to μ is $4811 - 2 \times 12{,}980{,}000\mu > 0$, indicating that $\mathbb{E}(\tau')$ is an increasing function of negative μ. In other words, as expected, increasing the *magnitude* of the negative number μ (or equivalently, decreasing μ) would decrease the expected liquidation time. The negative second-order partial derivative with respect to μ indicates for $\mu < 0$, $\mathbb{E}(\tau')$ is a *decelerated increasing function* of μ.

The effect of volatility-related variables $(\sigma, \sigma^{0.5}, \sigma^2)$ on $\mathbb{E}(\tau')$ is mixed. Intuitively, for high leverage positions with liquidation price close to their entry price, high volatility increases the risk to have an early liquidation. However, volatility also works to move the index price in the other direction further from liquidation. To see which scenario has a stronger effect, we observe $\forall \sigma > 0, \frac{\partial \mathbb{E}(\tau')}{\partial \sigma} = 1587 - 275.9\sigma^{-0.5} - 8012\sigma < 0$. The only critical point is the boundary value at $\sigma = 0$, where $\mathbb{E}(\tau')$ doesn't not depend on σ. Thus, $\mathbb{E}(\tau')$ decreases as σ increase, higher volatility decreases the expected liquidation time. Next, observe that $\frac{\partial^2 \mathbb{E}(\tau')}{\partial \sigma^2} = 137.95\sigma^{-1.5} - 8012$, $\frac{\partial^2 \mathbb{E}(\tau')}{\partial \sigma^2} > 0 \; if \; \sigma < 0.0667, and \; \frac{\partial^2 \mathbb{E}(\tau')}{\partial \sigma^2} < 0 \; if \; \sigma > 0.0667$. This shows that $\mathbb{E}(\tau')$'s decrease gets slower when σ_i approaches 0.0667, and gets faster when σ is further from 0.0667. Recall that our approximation for historical σ in **Table 10** of the Set Up section is 0.0005, much smaller than 0.0667, so the result indicates that for $\sigma \in [0, 0.0667]$, $\mathbb{E}(\tau')$ is a *decelerated decreasing function* of σ.

Table 11. Regression Coefficients

Variable	Coefficient
$intercept$	72.18 ***
	(43.909)
σ	1,587 ***
	(9.010)
$\sigma^{0.5}$	-551.8 ***
	(-15.676)
σ^2	$-4{,}006$ ***
	(-4.912)
μ	4,811 ***
	(3.599)

μ^2	$-12{,}980{,}000$ ***
	(-3.920)
L	-0.2208 ***
	(-16.095)
Sample	440
Adjusted R^2	0.8383

Note: t values are inside the parenthesis; *, **, *** means <5%, <1%, <0.1% significance level

3.4.3 Probability of Liquidation Before T: $\mathbb{P}(\tau' \leq T)$

The regression model:

$$\mathbb{P}(\tau' \leq T)_i = \beta_0 + \beta_1\sigma_i^{0.5} + \beta_2\sigma_i^2 + \beta_3\mu_i + \beta_4\mu_i^2 + \beta_5 L_i + \epsilon_i, \ \ i = 1, \dots, 440$$

The probability of getting liquidated before a fixed maturity $T = 80$ follows:

$$\mathbb{P}(\tau' \leq T)_i = 0.2686 + 2.531\sigma_i^{0.5} - 27.55\sigma_i^2 - 277.3\mu_i + 199300\mu_i^2 + 0.002945L_i + \epsilon_i$$
$$i = 1, \dots, 440$$

Table 12 summarizes the regression results; see Appendix 6 for the R report. The p-value (p-value: $< 2.2 \times 10^{-16}$) is much smaller than 0.01, thus there is significant linear relationship between the variables in the regression.

The coefficient for L is positive, indicating that the probability of getting liquidated before T increases as leverage increases.

As mentioned before, S_t tends to move downward for drift $\mu < 0$, for which the probability of getting liquidated in finite time is 1; in our model, for $\mu < 0$, the terms $\beta_3\mu + \beta_4\mu^2 > 0$, and the negative first-order partial derivative with respect to μ is $-277.3 + 199{,}300\mu < 0$, indicating that $\mathbb{P}(\tau' \leq T)$ is an decreasing function of negative μ. In other words, increasing the *magnitude* of the negative number μ (or equivalently, decreasing μ) would increase the probability of liquidation before T. The positive second-order partial derivative with respect to μ indicates that for $\mu < 0$, $\mathbb{P}(\tau' \leq T)$ is a *decelerated decreasing function* of μ. This matches with our previous findings that $\mathbb{E}(\tau')$ is a *decelerated increasing function* of μ.

To single out the effects of volatility-related terms $(\sigma^{0.5}, \sigma^2)$, we observe that $\forall\sigma \in (0, 0.0808)$, $\frac{\partial\mathbb{P}(\tau' \leq T)}{\partial\sigma} = 1.2655\sigma^{-0.5} - 55.1\sigma > 0, and\ \forall\sigma \in (0.0808, \infty), \frac{\partial\mathbb{P}(\tau' \leq T)}{\partial\sigma} = 1.2655\sigma^{-0.5} - 55.1\sigma < 0$. The two critical points are at $\sigma = 0$ and 0.0808. Next, $\frac{\partial^2\mathbb{P}(\tau' \leq T)}{\partial\sigma^2} = -0.63275\sigma^{-1.5} - 55.1 < 0, \forall\sigma > 0$, indicating that $\forall\sigma \in (0, 0.0808)$, $\mathbb{P}(\tau' \leq T)$ is an *decelerated increasing function* of σ, and $\forall\sigma \in (0.0808, \infty)$, $\mathbb{P}(\tau' \leq T)$ is an *accelerated decreasing function* of σ. The regression in **Table 10** of the Set Up section suggests that $\sigma \approx 0.0005$. For $\sigma \approx 0.0005$,

$\mathbb{P}(\tau' \leq T)$ is an *accelerated increasing function* of σ, indicating that increasing volatility would increase the probability of getting liquidated before T at a decreasing pace. This matches with our previous findings that $\mathbb{E}(\tau')$ is a *decelerated decreasing function* of σ.

Table 12. Regression Coefficients

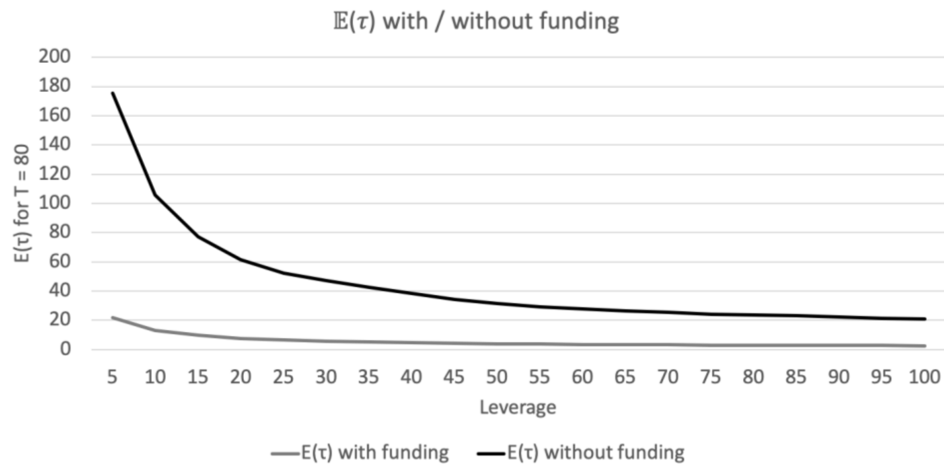
Variable	**Coefficient**
$intercept$	0.2686 ***
	(10.315)
$\sigma^{0.5}$	2.531 ***
	(15.327)
σ^2	-27.55 ***
	(-6.055)
μ	-277.3 ***
	(-9.663)
μ^2	$199{,}300$ **
	(2.789)
L	0.002945 ***
	(9.897)
Sample	440
Adjusted R^2	0.5924

Note: t values are inside the parenthesis; *, **, *** means <5%, <1%, <0.1% significance level

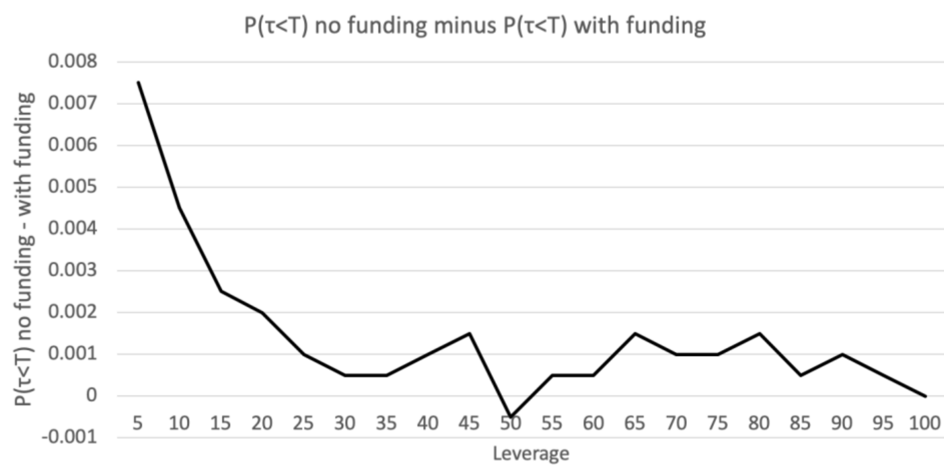
3.5 Should I Worry about Funding Rates?

Our results show that when funding is present, *high leverage* increases liquidation risks by scaling up the volatility of the cash flows. Also, based on simulations (unspecified in this paper), we postulate that *higher volatility of funding rates* also increase liquidation risks. Comparing our theoretical models earlier in this section, the liquidation price is constant in the model ignoring funding and fees (funding and fees are set to 0) and variable in the model considering funding and fees. Therefore, we postulate that with more volatile funding rates, the increasing risks of getting liquidated would transform into lower expected liquidation time and higher probability of liquidation before a certain time.

Using the same simulation paths for S_t, we compared the simulated expected liquidation time for liquidations before time T and the probability of liquidation before time T. **Figure 11** shows an example of such simulation; the example in general matches with our expectation that funding decreases the expected liquidation time and increases the probability of liquidation.



a.



b.

Figure 11. Comparison of simulated a. $\mathbb{E}(\tau)$ and b. $\mathbb{P}(\tau < T)$ no funding $-$ $\mathbb{P}(\tau < T)$ with funding ($\sigma = 0.05$ and $\mu = 0.0005$ for S_t)

4 Conclusion

In this study, we have analyzed multiple aspects of the BitMEX Perpetual Inverse Futures XBTUSD Contract. First, we provided a quantitative description of holding a long position of the contract. Then, we offered a regression model to characterize the funding rate as a function of index price and the Dow Jones Industrial Average. Next, we studied the effects of leverage on liquidation as a stopping time problem. To do so, we derived theoretical formulas for expected liquidation time and the probability of getting liquidated before a certain future time by using stochastic calculus methods, simulations, and regressions. In addition, we analyzed the effects of volatility and drift in the geometric Brownian motion model for index price and the leverage on the expected liquidation time and the probability of liquidation.

The results of this study will be useful for both investors and researchers in the finance field. For investors to assess their risks and design optimal trading strategies, the simulation code we provided in the appendix could be customized with different values according to the current cryptocurrency market. The systematic quantitative description of the XBTUSD contract would also provide researchers with further information into understanding the designing of cryptocurrency derivative exchanges. Furthermore, the theoretical models and regression models for the characteristics of liquidation times would also assist quantitative finance researchers towards building more realistic models in their future researches on Perpetual Inverse Futures.

Further relevant topics of interest include quantitatively describe the factors that determine the index price or the Bitcoin / USD exchange rates over major exchanges. Also, when funding and fees are considered, we have only studied the characteristics of leverage on liquidation in a fixed time period and under the same set of parameters for funding rates. In order to understand the effects of funding rate properties on liquidation and the distribution of liquidations over various timespans, more simulations needed to be tested. Finally, though few studies have been able to provide closed form solutions for first-hitting time problems on double stochastic processes, it is still a promising topic in the field of quantitative finance.

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Appendix

Appendix 1Derivation of Bankruptcy Price

Bankruptcy price is the spot price where "all initial margin is lost", thus

$$-\,Initial\ Margin = PnL = \#\ Contracts\ \times\ Multiplier\ \times \left(\frac{1}{Entry} - \frac{1}{Exit\ Price}\right)$$

Here $Exit\ Price = Bankruptcy\ Price$, so we get

$$-M_0 = NL \times 1 \times \left(\frac{1}{S_0} - \frac{1}{K}\right) = -\frac{N}{S_0}$$

therefore,

$$K(\$) = S_0 \times \frac{L}{L+1} = \frac{S_0 L}{L+1}$$

which means the bankruptcy price is $\$\,\frac{S_0 L}{L+1} = Ƀ1, or\ Ƀ\,\frac{L+1}{S_0 L} = \$\ 1$

Since we have NL contracts in total, at bankruptcy, our position is worth $\$\ NL = Ƀ\ N\,\frac{L+1}{S_0}$

Appendix 2Liquidation Definition

The stopping time τ is when our position gets liquidated by BitMEX. It happens when the margin balance goes below the maintenance margin requirement.

Margin Balance = Wallet Balance + Unrealised PNL.

Unrealised PNL = Current profit and loss from all open positions.

Wallet Balance = Deposits - Withdrawals + Realised PNL

--- *Margin Term Reference, BitMEX*

where

$$Unrealised\ PNL = Ƀ\ NL\left(\frac{1}{S_0} - \frac{1}{S_t}\right)$$

Appendix 3Funding Rate Regression R Output

Model 1 $lagTime = 0$

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0032752 -0.0001460 -0.0000015  0.0001728  0.0029876
##
## Coefficients:
```

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0015614  0.0001668  -9.359  < 2e-16 ***
## V2          -0.1460681  0.0269957  -5.411 7.36e-08 ***
## V3          14.2736774  1.8184213   7.849 8.18e-15 ***
## V4          68.6675722  6.2175112  11.044  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0003987 on 1416 degrees of freedom
## Multiple R-squared:  0.2725, Adjusted R-squared:  0.2709
## F-statistic: 176.8 on 3 and 1416 DF,  p-value: < 2.2e-16
```

Model 2 $lagTime = 0$ & without Dow Jones Term

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0035850 -0.0001504  0.0000313  0.0001614  0.0028875
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0000164  0.0000947  -0.173 0.862550
## V2          -0.0939510  0.0276915  -3.393 0.000711 ***
## V3          12.1822660  1.8841541   6.466 1.38e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0004154 on 1417 degrees of freedom
## Multiple R-squared:  0.2098, Adjusted R-squared:  0.2087
## F-statistic: 188.1 on 2 and 1417 DF,  p-value: < 2.2e-16
```

Model 3 $lagTime = 0$ & without Index Price Term

$$Y_t = \beta_0 + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0033219 -0.0001416 -0.0000288  0.0001707  0.0029382
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0018999  0.0001562  -12.16   <2e-16 ***
## V2           4.5889545  0.3240266   14.16   <2e-16 ***
## V3          62.7869114  6.1825618   10.15   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0004027 on 1417 degrees of freedom
```

```
## Multiple R-squared:  0.2574, Adjusted R-squared:  0.2564
## F-statistic: 245.6 on 2 and 1417 DF,  p-value: < 2.2e-16
```

Model 4 $lagTime = 0$ & without Index Price Squared Term

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_3 x_{3,t} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0033394 -0.0001374 -0.0000471  0.0001693  0.0029264
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.002104   0.000155  -13.57   <2e-16 ***
## V2           0.062510   0.004864   12.85   <2e-16 ***
## V3          63.585184   6.314580   10.07   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0004071 on 1417 degrees of freedom
## Multiple R-squared:  0.2408, Adjusted R-squared:  0.2398
## F-statistic: 224.8 on 2 and 1417 DF,  p-value: < 2.2e-16
```

Model 5 $lagTime = 1$

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t-1} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0032807 -0.0001471 -0.0000009  0.0001744  0.0029781
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.001568   0.000167  -9.388  < 2e-16 ***
## V2          -0.203602   0.065348  -3.116  0.00187 **
## V3          14.319741   1.821682   7.861 7.52e-15 ***
## V4          68.929468   6.220345  11.081  < 2e-16 ***
## V5           0.057076   0.058172   0.981  0.32668
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0003987 on 1414 degrees of freedom
## Multiple R-squared:  0.2735, Adjusted R-squared:  0.2714
## F-statistic: 133.1 on 4 and 1414 DF,  p-value: < 2.2e-16
```

Model 6 $lagTime = 2$

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + {}_4 x_{4,t-1} + \beta_5 x_{5,t-2} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0034854 -0.0001454 -0.0000037  0.0001682  0.0030650
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0015917  0.0001657  -9.606  < 2e-16 ***
## V2          -0.2272070  0.0649447  -3.498 0.000482 ***
## V3          13.6412783  1.8100621   7.536 8.60e-14 ***
## V4          68.7347074  6.1665520  11.146  < 2e-16 ***
## V5           0.3904060  0.0865116   4.513 6.93e-06 ***
## V6          -0.3003542  0.0580867  -5.171 2.67e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0003951 on 1412 degrees of freedom
## Multiple R-squared:  0.2875, Adjusted R-squared:  0.285
## F-statistic:   114 on 5 and 1412 DF,  p-value: < 2.2e-16
```

Model 7 $lagTime = 3$

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \beta_3 x_{3,t} + \beta_4 x_{4,t-1} + \beta_5 x_{5,t-2} + \beta_6 x_{6,t-3} + \epsilon_t$$

```
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0034392 -0.0001447 -0.0000025  0.0001633  0.0034335
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.001606   0.000164  -9.793  < 2e-16 ***
## V2          -0.208623   0.064292  -3.245   0.0012 **
## V3          13.033724   1.792857   7.270 5.94e-13 ***
## V4          68.311886   6.099531  11.200  < 2e-16 ***
## V5           0.342342   0.085955   3.983 7.16e-05 ***
## V6           0.072035   0.085942   0.838   0.4021
## V7          -0.334606   0.057414  -5.828 6.94e-09 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0003907 on 1410 degrees of freedom
## Multiple R-squared:  0.3045, Adjusted R-squared:  0.3015
## F-statistic: 102.9 on 6 and 1410 DF,  p-value: < 2.2e-16
```

Appendix 4Simulation Codes

Simulation Codes with a set of sample parameters

```
import numpy as np
import matplotlib.pyplot as plt

### Constants
T = 80                # Total number of time periods
dt = 0.125            # The rate of sampling in terms of time periods.
du = 1                # The total number of samples is T/dt
numPaths = 2000       # simulate numPath number of paths
N = 1                 # number of contracts
S0 = 10000            # initial index price for every path
r0 = 0.0001           # initial funding rate for every path
cap = 0.0045          # funding rate cap

## Index price parameters
sigma = 0.0005
mu = -0.0001

## Funding rate parameters
b1 = 0.00003          # (1-rho) * zeta
b2 = 0.75             # rho
b3 = -0.0003           # (1-rho^2 )^0.5 * sigma_r

### Leverage
np.random.seed(1)
leverage = [i*5 for i in range(1,21)]       # leverages to test

### Reports
SimulationResultNFFile = "/Users/sarah/Desktop/SimulationResultNoFunding.txt"
SimulationResultFile = "/Users/sarah/Desktop/SimulationResult.txt"

###### INDEX PRICE SIMULATION
# paths
def indexPricePath(sigma, mu):
    S_path = []                             # paths of S = [path1, path2, ...]
    for i in range (numPaths):
        newPath = [S0]
        randEpsilon = np.random.normal(0, np.sqrt(dt), int(T/dt))
        for j in range (int(T/dt)):
            temp = newPath[-1] * (np.exp((mu - sigma ** 2 / 2) * dt + sigma * randEpsilon[j]))
            newPath.append(temp)
        S_path.append(newPath)              # newPath = [S0,S1,S2,...,S(T/dt)]
    return S_path

# plot
def indexPricePathPlot(S_path):
    x = [i for i in range(int(T/dt)+1)]
    for i in range(numPaths):
        plt.plot(x, S_path[i])
    plt.xlabel("$t$")
    plt.ylabel("$S_t$")
    #plt.ylim(800,20000)
    plt.title("Geometric Brownian Motion Index Price $S_t$\n")
    plt.show()

###### FUNDING RATE SIMULATION
```

```
# paths
def fundingRatePath(b1, b2, b3, r0):
    r_path = []                             # paths of r = [path1, path2, ...]
    for i in range (numPaths):
        newPath = [r0]
        lastRate = r0
        randEpsilon = np.random.normal(0, np.sqrt(du), int(T/du))
        count = 0
        fundingPeriod = du / dt             # number of dt per du
        for j in range (int(T/dt)):
            if j % fundingPeriod != 0:      # funding only happens at every funding timestamp
                newPath.append(0)
            else:
                temp = b1 + lastRate * b2 + b3 * randEpsilon[count]
                count += 1
                if temp > cap:   temp = cap
                elif temp < -cap:    temp = -cap
                lastRate = temp
                newPath.append(temp)
        r_path.append(newPath)              # newPath = [r0,r1,r2,...,r(T/du)]
    return r_path

# plot
def fundingRatePathPlot(r_path):
    x1 = [i for i in range(int(T/dt)+1)]
    for i in range(numPaths):
        plt.plot(x1, r_path[i])
    plt.xlabel("$t$")
    plt.ylabel("$r_t$")
    plt.ylim(-0.01,0.01)
    plt.title("Mean-Reverting Funding Rate $r_t$\n")
    plt.show()

##### Liquidation - With Funding
def lqPrc(allS,allR,crrntIndex, L):
    numerator = (L ** 2) * S0
    k = 0
    for i in range (1,crrntIndex):
        k += allR[i] * S0 / allS[i]
    denominator = (L ** 2) * (1 - k)
    denominator += L * (0.99525 - allR[crrntIndex])
    denominator -= 0.00075
    denominator -= allR[crrntIndex]
    return (numerator / denominator)

def withFunding(L):
    M0 = N / S0                             # initial margin
    # Expected Liquidation Time E(τ)
    lqPrice = 0
    lqTime = []
    lq = 0                                  # number of path that are being liquidated before T
    for path in range(numPaths):
        # for each path
        for i in range (1, int(T/dt + 1)):
            lqPrice = lqPrc(S_path[path],r_path[path],i,L)
            if S_path[path][i] < lqPrice:
                lqTime.append(i*dt)
                lq += 1
                break
```

```
        ExpLqTime = np.average(lqTime)      # if τ <= T, the expected τ
        # Probability of getting liquidated (at least once) in T time periods
        LqProb = lq / numPaths
        return (ExpLqTime, LqProb, lq, lqTime)

    # Results File
    def writeFunding():
        ETau = []
        tt = "With Funding"
        tt += "\nNumber of Margin Contracts: N = " + str(N)
        tt += "\nInitial Index Price: S0 = " + str(S0)
        tt += "\nInitial Funding Rate: r0 = " + str(r0)
        tt += "\nb1 = " + str(b1) + "\nb2 = " + str(b2) + "\nb3 = " + str(b3)
        tt += "\n\nTotal Time Periods: T = " + str(T)
        tt += "\n\nTotal Hours: 8T/du = " + str(8*T/du)
        tt += "\nSampling Time Period: dt = " + str(dt)
        tt += "\nNumber of Funding Timestamps: T/du = " + str(T/du)
        tt += "\nNumber of Path Simulated = " + str(numPaths)
        for i in range (len(leverage)):
            temp = withFunding(leverage[i])
            addtt = "\n\nLeverage: L = " + str(leverage[i])
            addtt += "\nExpected Liquidation Time = " + str(temp[0])
            addtt += "\nProbability of Liquidation on or before hour " + str(T / du) + " = " + str (temp[1])
            ETau.append(temp[0])
            tt += addtt

        f = open(SimulationResultFile,"w+")
        with open(SimulationResultFile, "wt") as f:
            f.write(tt)

        x1 = leverage
        for i in range(len(leverage)):
            plt.plot(x1, ETau)
        plt.xlabel("$leverage$")
        plt.ylabel("$E(τ)$")
        plt.xlim(0,100,5)
        plt.ylim(0,T)
        plt.title("$E(τ)$ verses leverage")
        plt.show()

        # histogram of the distribution of all liquidation times
        n, bins, patches = plt.hist(temp[3], temp[2], density=True, facecolor='r', alpha=0.75)
        plt.ylabel("frequency")
        plt.xlabel("time period")
        plt.title("Distribution of Liquidation Times for "+str(temp[2])+" liquidations over "+str(numPaths)+" simulations \n Leverage = "+str(leverage[i]))
        plt.show()

    ##### Liquidation - NO Funding
    def noFunding(L):
        lqPriceNF = L * S0 / (L + 0.996)
        # Expected Liquidation Time E(τ)
        lqTimeNF = []
        lqNF = 0        # number of path that are being liquidated before T
        for path in range(numPaths):
            for i in range (1, int(T/dt + 1)):
                if S_path[path][i] < lqPriceNF:
                    lqTimeNF.append(i)
                    lqNF += 1
```

```
                    break
            EpdLqTimeNF = np.average(lqTimeNF)
            # Probability of Liquidation
            # Probability of getting liquidated at least once in T time periods
            LqProbNF = lqNF / numPaths
            return (lqPriceNF, EpdLqTimeNF, LqProbNF, lqNF, lqTimeNF)

        # Results File
        def writeNoFunding():
            Etau = []
            tt = "Without Funding"
            tt += "\nNumber of Margin Contracts: N = " + str(N)
            tt += "\nInitial Index Price: S0 = " + str(S0)
            tt += "\nsigma = " + str(sigma) + "\nmu = " + str(mu)
            tt += "\n\nTotal Time Periods: T = " + str(T)
            tt += "\n\nTotal Hours: 8T/du = " + str(8*T/du)
            tt += "\nSampling Time Period: dt = " + str(dt)
            tt += "\nNumber of Path Simulated = " + str(numPaths)
            for i in range (len(leverage)):
                temp = noFunding(leverage[i])
                addtt = "\n\nLeverage: L = " + str(leverage[i])
                addtt += "\nLiquidation Price = " + str(temp[0])
                addtt += "\nExpected Liquidation Time = " + str(temp[1])
                addtt += "\nProbability of Liquidation on or before hour " + str(T / du) + " = " + str (temp[2])
                tt += addtt
                Etau.append(temp[1])
            f = open(SimulationResultNFFile,"w+")
            with open(SimulationResultNFFile, "wt") as f:
                f.write(tt)

            x1 = leverage
            for i in range(len(leverage)):
                plt.plot(x1, Etau)
            plt.xlabel("$leverage$")
            plt.ylabel("$E(τ)$")
            plt.xlim(0,100,5)
            plt.ylim(0,T)
            plt.title("$E(τ)$ verses leverage")
            plt.show()

            # histogram of the distribution of all liquidation times
            n, bins, patches = plt.hist(temp[4], temp[3], density=True, facecolor='b', alpha=0.75)
            plt.ylabel("frequency")
            plt.xlabel("time period")
            plt.title("Distribution of Liquidation Times for "+str(temp[3])+" liquidations over "+str(numPaths)+" simulations \n Leverage = "+str(leverage[i]))
            plt.show()

        ##### Run Program
        S_path = indexPricePath(sigma, mu)
        indexPricePathPlot(S_path)
        r_path = fundingRatePath(b1, b2, b3, r0)
        fundingRatePathPlot(r_path)
        writeNoFunding()
        writeFunding()
```

Appendix 5 $\mathbb{E}(\tau')$ Regression R Output

$$\mathbb{E}(\tau')_i = \beta_0 + \beta_1\sigma_i + \beta_2\sigma_i^{0.5} + \beta_3\sigma_i^2 + \beta_4\mu_i + \beta_5\mu_i^2 + \beta_6 L_i + \epsilon_i, \quad i = 1, \dots, 440$$

```
## Residuals:
##     Min      1Q  Median      3Q     Max
## -27.112  -5.393  -0.978   4.439  39.182
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  7.218e+01  1.644e+00  43.909  < 2e-16 ***
## V2           1.587e+03  1.761e+02   9.010  < 2e-16 ***
## V3          -5.518e+02  3.520e+01 -15.676  < 2e-16 ***
## V4          -4.006e+03  8.156e+02  -4.912 1.28e-06 ***
## V5           4.811e+03  1.337e+03   3.599 0.000357 ***
## V6          -1.298e+07  3.312e+06  -3.920 0.000103 ***
## V7          -2.208e-01  1.372e-02 -16.095  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 7.998 on 432 degrees of freedom
## Multiple R-squared:  0.8405, Adjusted R-squared:  0.8383
## F-statistic: 379.5 on 6 and 432 DF,  p-value: < 2.2e-16
```

Appendix 6 $\mathbb{P}(\tau' \leq T)$ Regression R Output

$$\mathbb{P}(\tau' \leq T)_i = \beta_0 + \beta_1\sigma_i^{0.5} + \beta_2\sigma_i^2 + \beta_3\mu_i + \beta_4\mu_i^2 + \beta_5 L_i + \epsilon_i, \quad i = 1, \dots, 440$$

```
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.64658 -0.07983  0.02802  0.11055  0.42036
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.686e-01  2.604e-02  10.315  < 2e-16 ***
## V2           2.531e+00  1.652e-01  15.327  < 2e-16 ***
## V3          -2.755e+01  4.550e+00  -6.055 3.04e-09 ***
## V4          -2.773e+02  2.870e+01  -9.663  < 2e-16 ***
## V5           1.993e+05  7.148e+04   2.789  0.00553 **
## V6           2.945e-03  2.976e-04   9.897  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.1735 on 433 degrees of freedom
## Multiple R-squared:  0.597,  Adjusted R-squared:  0.5924
## F-statistic: 128.3 on 5 and 433 DF,  p-value: < 2.2e-16
```