



**Illinois Wesleyan University**  
**Digital Commons @ IWU**

---

John Wesley Powell Student Research  
Conference

2013, 24th Annual JWP Conference

---

Apr 20th, 9:00 AM - 10:00 AM

## Frames and Spline Framelets

Tung Nguyen  
*Illinois Wesleyan University*

Nahee Kim  
*Illinois Wesleyan University*

Tian-Xiao He, Faculty Advisor  
*Illinois Wesleyan University*

Follow this and additional works at: <https://digitalcommons.iwu.edu/jwprc>



Part of the [Mathematics Commons](#)

---

Nguyen, Tung; Kim, Nahee; and He, Faculty Advisor, Tian-Xiao, "Frames and Spline Framelets" (2013). *John Wesley Powell Student Research Conference*. 15.  
<https://digitalcommons.iwu.edu/jwprc/2013/posters/15>

This Event is protected by copyright and/or related rights. It has been brought to you by Digital Commons @ IWU with permission from the rights-holder(s). You are free to use this material in any way that is permitted by the copyright and related rights legislation that applies to your use. For other uses you need to obtain permission from the rights-holder(s) directly, unless additional rights are indicated by a Creative Commons license in the record and/ or on the work itself. This material has been accepted for inclusion by faculty at Illinois Wesleyan University. For more information, please contact [digitalcommons@iwu.edu](mailto:digitalcommons@iwu.edu).

©Copyright is owned by the author of this document.



# Frames and Spline Framelets

Tung Nguyen, Nahee Kim and Tian-Xiao He\*

Mathematics Department, Illinois Wesleyan University

## Introduction

Frames can be seen as a generalization of the idea of orthonormal bases, which not only maintain useful characteristics of orthonormal bases but also allow more flexibility in applications.

The applications of frames include communication and image processing, as its characteristic inherited from orthonormal bases helps speed up the transmitting and processing time while its additional flexibility adds to frames the ability to reconstruct lost information.

In this project, we study the construction of a class of tight frames in Euclidean spaces  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

## Formal definitions

**Definition 1.** A frame for a Hilbert space  $\mathcal{H}$  is a sequence of vectors  $\{x_i\} \subset \mathcal{H}$  for which there exists constants  $0 < A \leq B < \infty$  such that, for every  $x \in \mathcal{H}$ ,

$$A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2.$$

The constants A and B are called respectively lower and upper *frame bounds*.

**Definition 2.** A tight frame for a Hilbert space  $\mathcal{H}$  is a sequence of vectors  $\{x_i\} \subset \mathcal{H}$  for which there exists constants  $0 < A$  such that, for every  $x \in \mathcal{H}$ ,

$$\sum_i |\langle x, x_i \rangle|^2 = A\|x\|^2.$$

A Parseval frame is a special case of tight frame when  $A = 1$ , or equivalently

$$\sum_i |\langle x, x_i \rangle|^2 = \|x\|^2.$$

**Definition 3.** Let  $\{x_i\}_{i=1}^k \subset \mathcal{H}$ . The operator  $\theta : \mathcal{H} \rightarrow \mathbb{R}^k$  defined by

$$\theta x = \begin{bmatrix} \langle x, x_1 \rangle \\ \vdots \\ \langle x, x_k \rangle \end{bmatrix} = \sum_{i=1}^k \langle x, x_i \rangle e_i$$

is called the analysis operator of  $\{x_i\}_{i=1}^k$ , where  $\{e_i\}_{i=1}^k$  is the standard orthonormal basis for  $\mathbb{R}^k$ .

## Important propositions

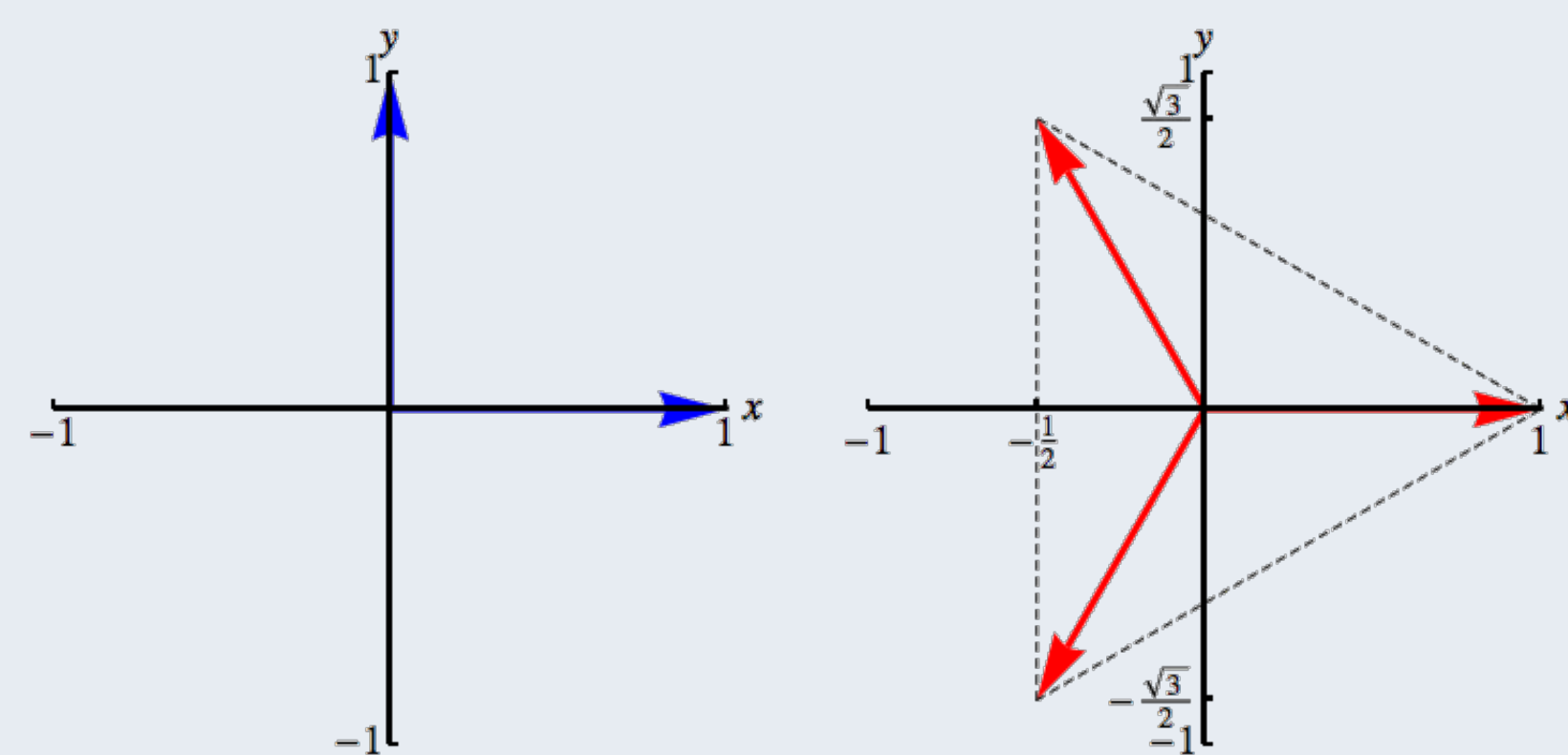
**Proposition 1.** A set of vectors  $\{x_i\}_{i=1}^k$  is a Parseval frame for a Hilbert space  $\mathcal{H}$  if and only if for every  $x$  in  $\mathcal{H}$  the following formula holds:  $x = \sum_{i=1}^k \langle x, x_i \rangle x_i$ .

**Proposition 2.** A set of vectors  $\{x_i\}_{i=1}^k$  is a frame for a finite-dimensional Hilbert space  $\mathcal{H}$  if and only if  $\{x_i\}_{i=1}^k$  is a spanning set of  $\mathcal{H}$ .

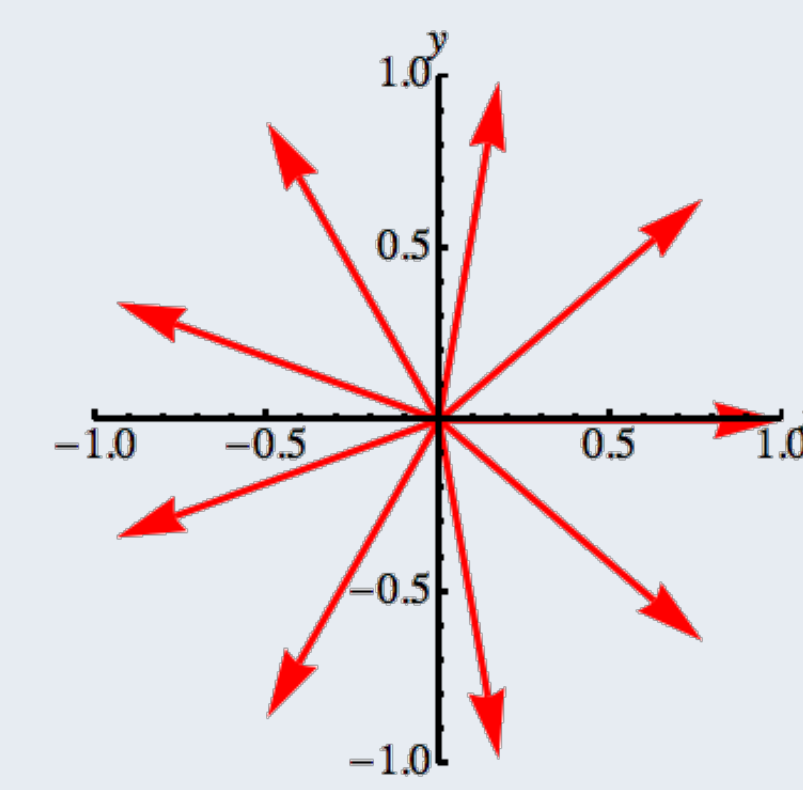
**Proposition 3.** Let  $\{x_i\}_{i=1}^k$  be a tight frame in  $\mathcal{H}$ . Then  $\theta^* \theta = AI$  where  $\theta$  is the analysis operator of  $\{x_i\}_{i=1}^k$  and  $\theta^*$  is the adjoint operator of  $\theta$ .

## Frames in $\mathbb{R}^2$

**Figure 1.** A typical tight frame with 3 vectors in  $\mathbb{R}^2$  and the standard orthonormal basis of  $\mathbb{R}^2$



**Figure 2.** A class of tight frame in  $\mathbb{R}^2$  with n vectors



## Construction of frames in $\mathbb{R}^3$

We use spherical coordinates to express any vector

$$x_i \text{ in } \mathbb{R}^3 \text{ in the form } x_i = \begin{bmatrix} a_i \sin \phi_i \cos \varphi_i \\ a_i \sin \phi_i \sin \varphi_i \\ a_i \cos \phi_i \end{bmatrix}$$

Five conditions that  $\{x_i\}_{i=1}^k$  must satisfy to be a tight frame in  $\mathbb{R}^3$

1.  $\sum_{i=1}^k a_i^2 \sin^2 \phi_i \sin \varphi_i \cos \varphi_i = 0$
2.  $\sum_{i=1}^k a_i^2 \sin \phi_i \cos \phi_i \cos \varphi_i = 0$
3.  $\sum_{i=1}^k a_i^2 \sin \phi_i \cos \phi_i \sin \varphi_i = 0$
4.  $\sum_{i=1}^k a_i^2 \sin^2 \phi_i \cos^2 \varphi_i = \sum_{i=1}^k a_i^2 \sin^2 \phi_i \sin^2 \varphi_i$
5.  $\sum_{i=1}^k a_i^2 \sin^2 \phi_i \sin^2 \varphi_i = \sum_{i=1}^k a_i^2 \cos^2 \phi_i$

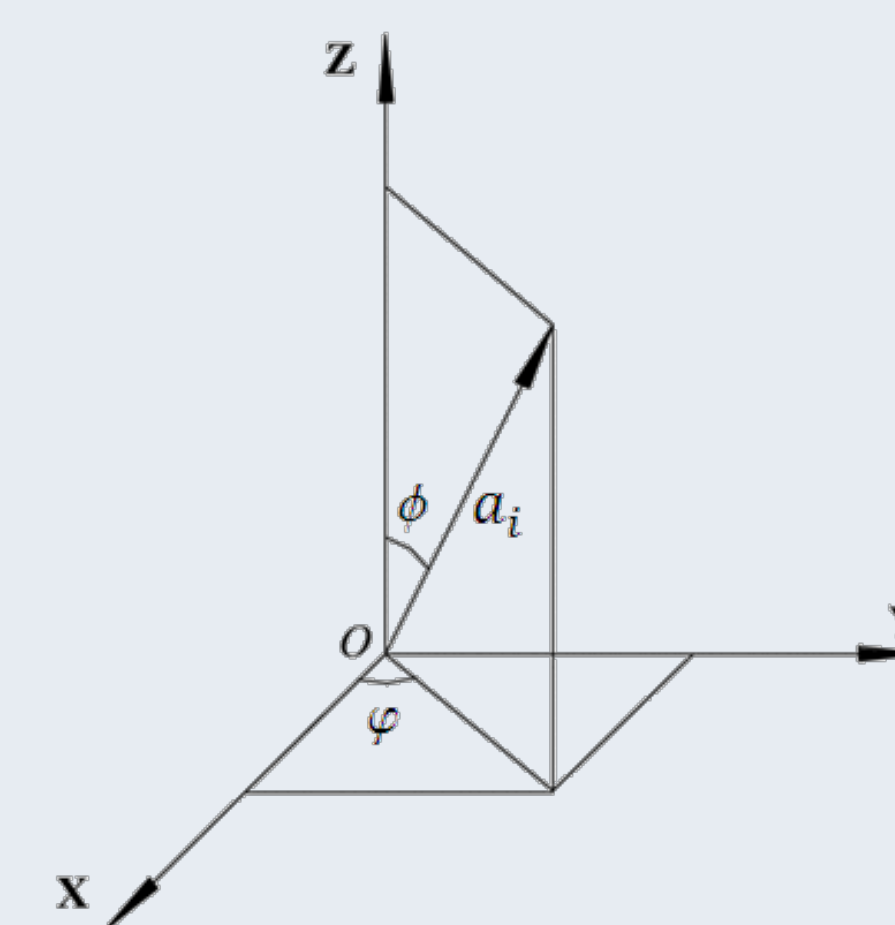
The class of tight frame we constructed

$$x_i = \begin{bmatrix} \sin \phi \cos \frac{2\pi(i-1)}{k-1} \\ \sin \phi \sin \frac{2\pi(i-1)}{k-1} \\ \cos \phi \end{bmatrix} \quad i = 1, \dots, k-1;$$

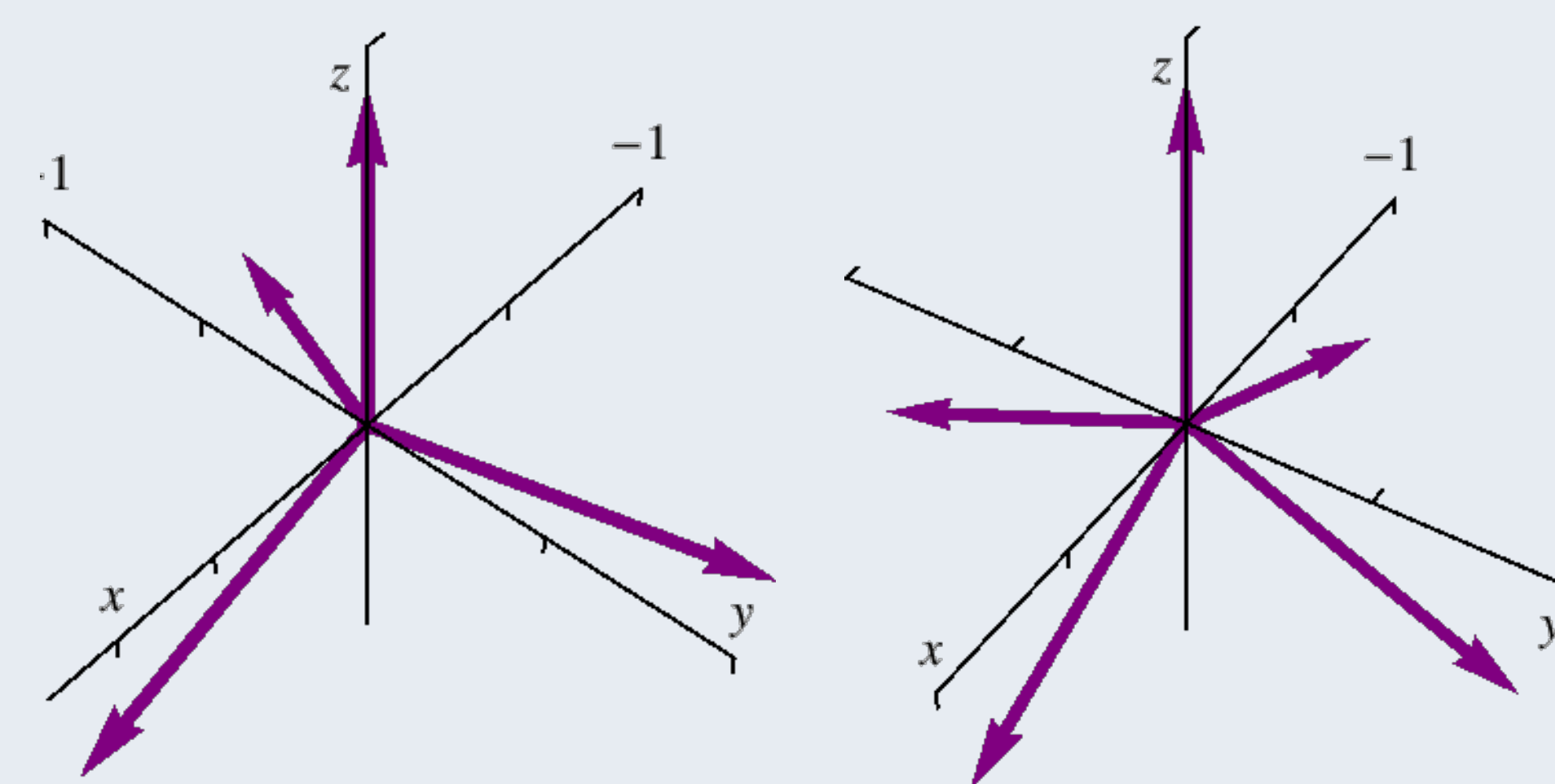
where  $k \geq 3$  and  $\phi = \arctan(\sqrt{\frac{2k}{k-3}})$  or

$$\pi - \arctan(\sqrt{\frac{2k}{k-3}}), \text{ and } x_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**Figure 3.** Parameterization of a vector in  $\mathbb{R}^3$



**Figure 4.** Illustrations of the class of frame we constructed - the cases k = 4 and k = 5.



## Further research

We want to expand our research interest to the construction of frames in not only Euclidean spaces but also other spaces. One space of special interest is the one generated by wavelet functions, which has a lot of applications in image processing.

Some of the literatures dealt with this topic, e.g., using Multi-resolution Analysis (MRA) and Fourier transforms to construct wavelet frames, or framelets. However, the Fourier transforms and window Fourier transforms are not flexible, which can not localize very high frequencies and very low frequencies. Therefore, there are still rooms to investigate, improve and expand the existed construction of framelets, in particular, spline framelets.

## References

- [1] D. Han, K. Kornelson, D. Larson, and E. Weber, Frames for Undergraduates, AMS, Rhode Island, 2007.
- [2] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14(2003), No.1, 1-46.

## About this research

This work was completed under the direction of **Professor Tian-Xiao He** as an individual project.

**Tung Nguyen**, *Mathematics & Economics*  
Illinois Wesleyan University  
tunguyen@iwu.edu

**Nahee Kim**, *Mathematics & Chemistry*  
Illinois Wesleyan University  
nkim@iwu.edu