Frames and Spline Framelets

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Introduction

Frames can be seen as a generalization of the idea of orthonormal bases, which not only maintain useful characteristics of orthonormal bases but also allow more flexibility in applications.

The applications of frames include communication and image processing, as its characteristic inherited from orthonormal bases helps speed up the transmitting and processing time while its additional flexibility adds to frames the ability to reconstruct lost information.

In this project, we study the construction of a class of tight frames in Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.

Formal definitions

Definition 1. *A frame for a Hilhert space $\mathcal{H}$ is a sequence of vectors $\{x_i\} \subset \mathcal{H}$ for which there exists constants $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,*

$$A\|x\|^2 \leq \Sigma_i |\langle x, x_i\rangle|^2 \leq B\|x\|^2.$$

The constants A and B are called respectively lower and upper *frame bounds*.

Definition 2. *A tight frame for a Hilhert space $\mathcal{H}$ is a sequence of vectors $\{x_i\} \subset \mathcal{H}$ for which there exists constants $0 < A$ such that, for every $x \in \mathcal{H}$,*

$$\Sigma_i |\langle x, x_i\rangle|^2 = A\|x\|^2.$$

A Parseval frame is a special case of tight frame when A = 1, or equivalently

$$\Sigma_i |\langle x, x_i\rangle|^2 = \|x\|^2.$$

Definition 3. *Let $\{x_i\}_{i=1}^k \subset \mathcal{H}$. The operator* $\theta : \mathcal{H} \to \mathbb{R}^k$ defined by

$$\theta x = \begin{bmatrix} \langle x, x_1\rangle \\ \cdot \\ \cdot \\ \langle x, x_k\rangle \end{bmatrix} = \Sigma_{i=1}^k \langle x, x_i\rangle e_i$$

is called the analysis operator of $\{x_i\}_{i=1}^k$, where $\{e_i\}_{i=1}^k$ is the standard orthonormal basis for $\mathbb{R}^k$.

Important propositions

Proposition 1. *A set of vectors $\{x_i\}_{i=1}^k$ is a Parseval frame for a Hilhert space $\mathcal{H}$ if and only if for every x in $\mathcal{H}$ the following formula holds:* $x = \sum_{i=1}^k \langle x, x_i\rangle x_i$.

Proposition 2. *A set of vectors $\{x_i\}_{i=1}^k$ is a frame for a finite-dimensional Hilhert space $\mathcal{H}$ if and only if $\{x_i\}_{i=1}^k$ is a spanning set of $\mathcal{H}$.*

Proposition 3. *Let $\{x_i\}_{i=1}^k$ be a tight frame in $\mathcal{H}$. Then $\theta^*\theta = AI$ where θ is the analysis operator of $\{x_i\}_{i=1}^k$ and θ^* is the adjoint operator of θ.*

Frames in $\mathbb{R}^2$

Figure 1. A typical tight frame with 3 vectors in $\mathbb{R}^2$ and the standard orthonormal basis of $\mathbb{R}^2$

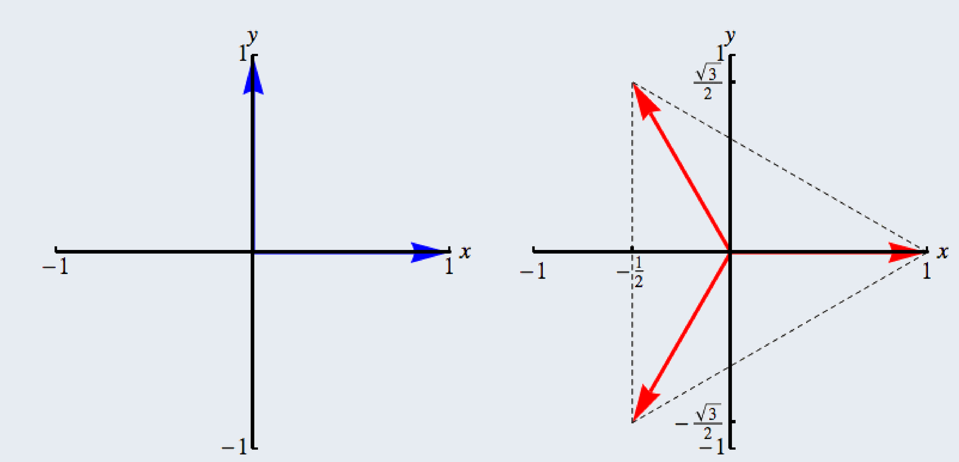
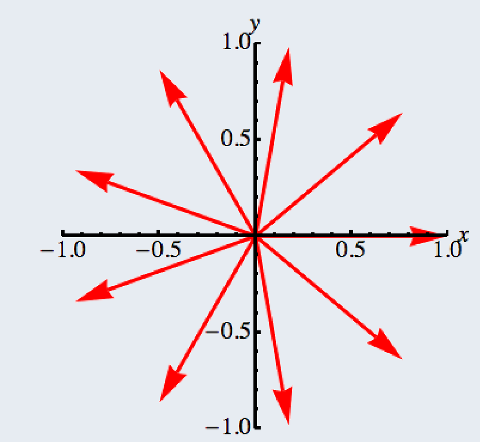


Figure 2. A class of tight frame in $\mathbb{R}^2$ with n vectors



Construction of frames in $\mathbb{R}^3$

We use spherical coordinates to express any vector x_i in $\mathbb{R}^3$ in the form $x_i = \begin{bmatrix} a_i \sin\phi_i \cos\varphi_i \\ a_i \sin\phi_i \sin\varphi_i \\ a_i \cos\phi_i \end{bmatrix}$

Five conditions that $\{x_i\}_{i=1}^k$ must satisfy to be a tight frame in $\mathbb{R}^3$

1. $\sum_{i=1}^k a_i^2 \sin^2\phi_i \sin\varphi_i \cos\varphi_i = 0$
2. $\sum_{i=1}^k a_i^2 \sin\phi_i \cos\phi_i \cos\varphi_i = 0$
3. $\sum_{i=1}^k a_i^2 \sin\phi_i \cos\phi_i \sin\varphi_i = 0$
4. $\sum_{i=1}^k a_i^2 \sin^2\phi_i \cos^2\varphi_i = \sum_{i=1}^k a_i^2 \sin^2\phi_i \sin^2\varphi_i$
5. $\sum_{i=1}^k a_i^2 \sin^2\phi_i \sin^2\varphi_i = \sum_{i=1}^k a_i^2 \cos^2\phi_i$

The class of tight frame we constructed

$$x_i = \begin{bmatrix} \sin\phi \cos\frac{2\pi(i-1)}{k-1} \\ \sin\phi \sin\frac{2\pi(i-1)}{k-1} \\ \cos\phi \end{bmatrix} \text{ i =1 ,..., k-1;}$$

where $k \geq 3$ and $\phi = arctan(\sqrt{\frac{2k}{k-3}})$ or $\pi - arctan(\sqrt{\frac{2k}{k-3}})$, and $x_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Figure 3. Parameterization of a vector in $\mathbb{R}^3$

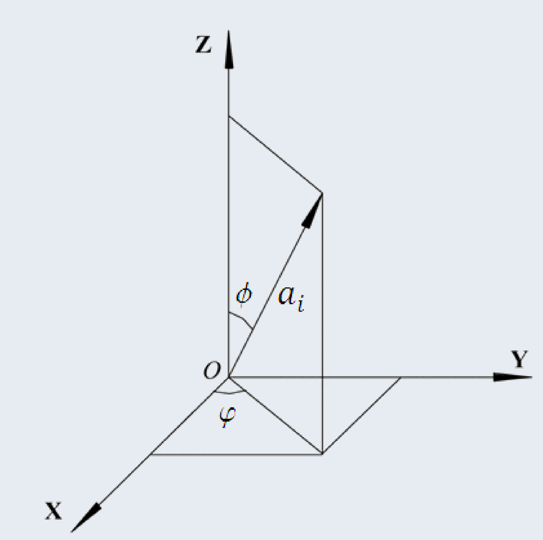
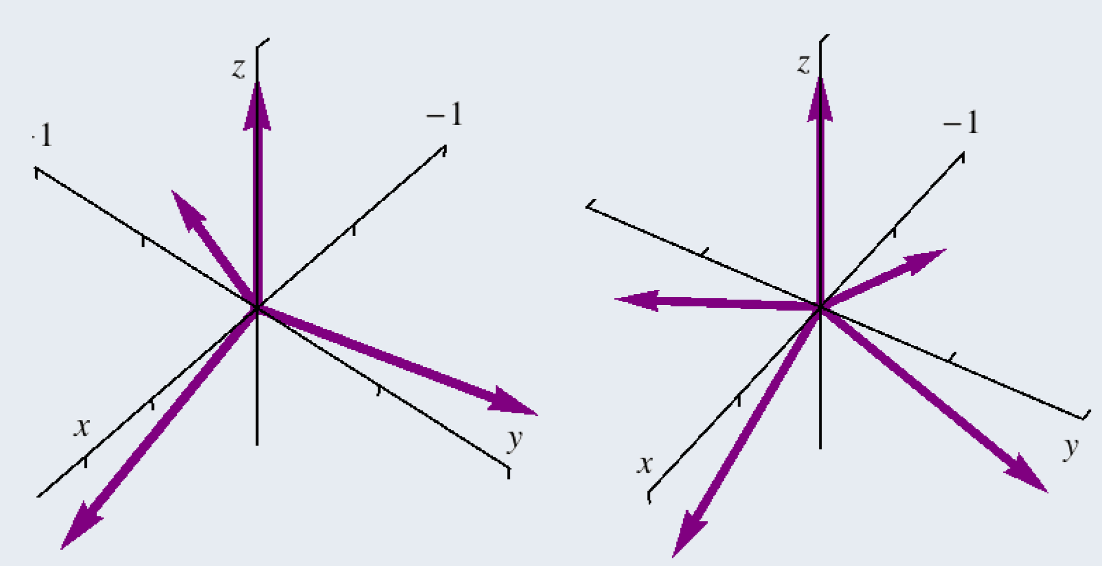


Figure 4. Illustrations of the class of frame we constructed - the cases k = 4 and k = 5.



Further research

We want to expand our research interest to the construction of frames in not only Euclidean spaces but also other spaces. One space of special interest is the one generated by wavelet functions, which has a lot of applications in image processing.

Some of the literatures dealt with this topic, e.g., using Multi-resolution Analysis (MRA) and Fourier transforms to construct wavelet frames, or framelets. However, the Fourier transforms and window Fourier transforms are not flexible, which can not localize very high frequencies and very low frequencies. Therefore, there are still rooms to investigate, improve and expand the existed construction of framelets, in particular, spline framelets.

References

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[2] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14(2003), No.1, 1-46.

About this research

This work was completed under the direction of **Professor Tian-Xiao He** as an individual project.

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