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In Pursuit of the Ringel-Kotzig Conjecture: Uniform K-Distant Trees are Graceful

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Uniform *k*-distant trees are graceful

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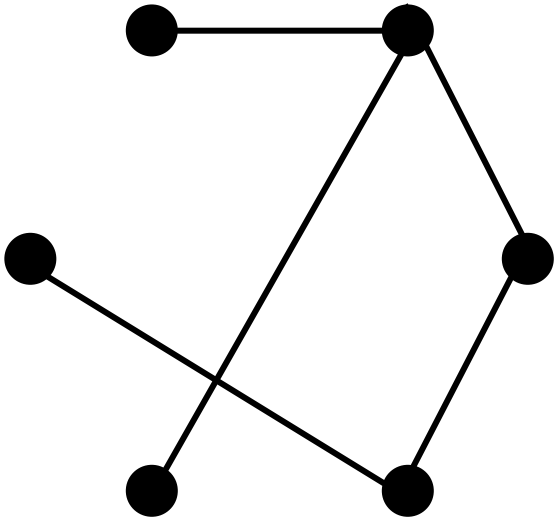
Brief Introduction to Graph Theory

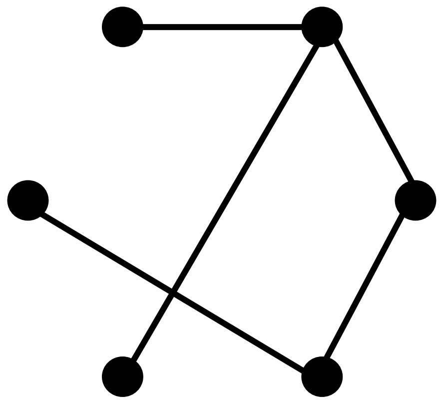
- A **graph** *G* consists of two sets:
  - *V(G), a set of vertices*



  - *And E(G), a set of edges*

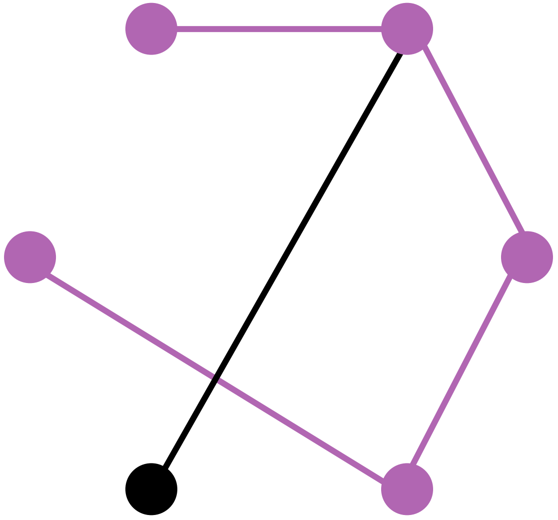
Brief Introduction to Graph Theory

- The **order** of a graph G is the number of vertices it has.
- The **size** of a graph G is the number of edges it has.
- The **degree** of a vertex is the number of edges incident to that vertex. A vertex of degree one is called a **leaf.**



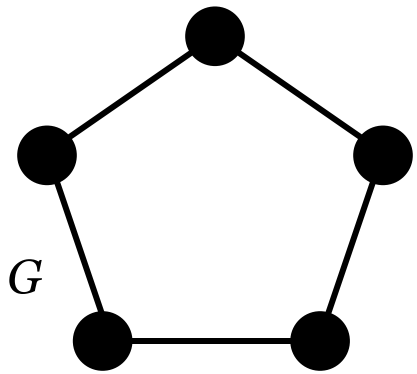
Brief Introduction to Graph Theory

- A **path** in a graph G is a sequence of distinct vertices such that there is an edge between each set of consecutive vertices.



Brief Introduction to Graph Theory

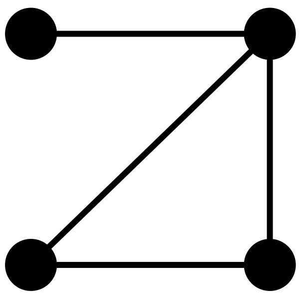
- A **cycle** in a graph *G* is a “path” that begins and ends with the same vertex.
- *G* is **connected** if and only if every pair of vertices can be joined by a path in *G*.
- If *G* is **acyclic,** then *G* has no cycles.



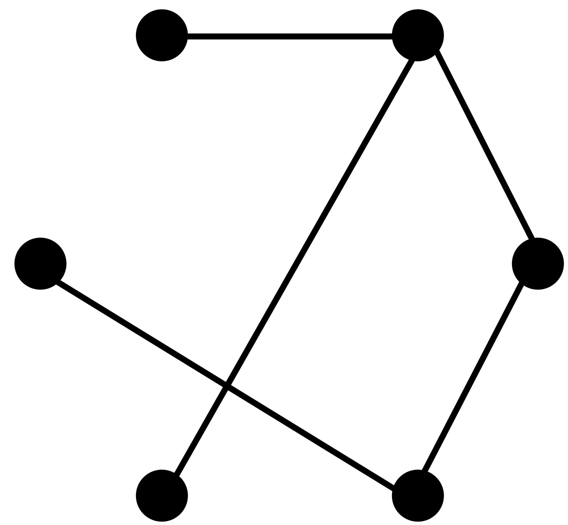
In this example, *G* is connected, but not acyclic.

Some important definitions

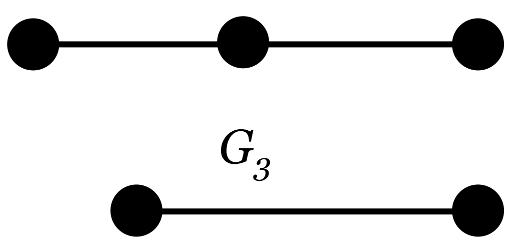
- A **tree** is a connected, acyclic graph.



G_1



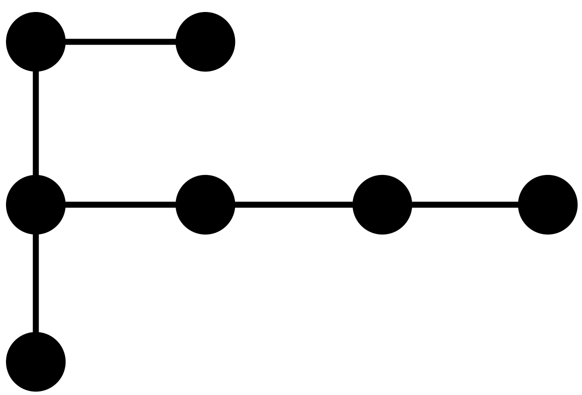
G_2

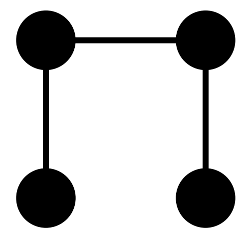


Some important definitions

- A ***k*-distant tree** consists of a main path, called the **spine**, such that each vertex on the spine is joined by an edge to an end-vertex of at most one path on at most *k* vertices.
- Those paths, along with the edge joining them to the spine, are called **tails**.

G_1

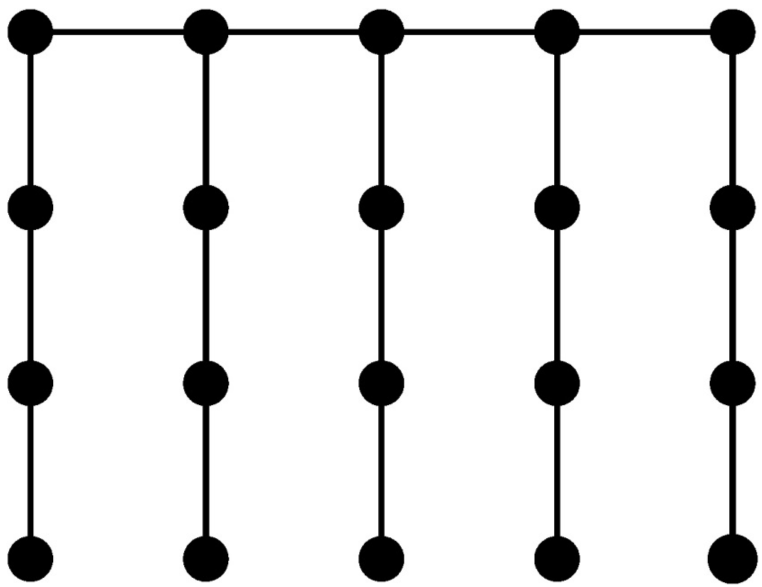




G_2

Some important definitions

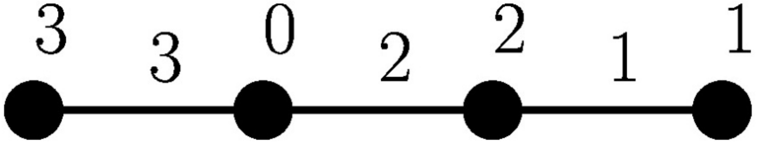
- When every vertex on the spine has exactly one incident tail of length *k* we call the tree a **uniform k-distant tree**.



A uniform 3-distant tree

Some more definitions

- A **graceful labeling** of a graph G on *n* vertices is a one-to-one function from the vertices of G to the set $\{0, \ldots, |E(G)|\}$ such that the induced edge labels given by $|f(u) - f(v)|$, for every uv in $E(G)$, are all distinct. If a graph admits a graceful labeling then that graph is said to be **graceful**.



An example of a graceful labeling

History

- Rosa first introduced graceful labelings in 1967 to study decompositions.
- While these labelings continue to be used in the study of decompositions, they have since become a thoroughly studied subject of their own.
- Soon after Rosa's introduction, almost 50 years ago, Ringel and Kotzig made the following conjecture, also referred to as the Graceful Tree Conjecture.

Conjecture 1. (Ringel-Kotzig)
All trees are graceful.

Progress

- In support of the conjecture, many classes of trees or trees with certain numbers of vertices have been shown to be graceful (Gallian).
- Of special interest to us are caterpillars and lobsters. A **caterpillar** is a tree of order three or more, the removal of whose degree one vertices produces a path. A **lobster** is a tree with the property that the removal of the degree one vertices leaves a caterpillar.
  - Uniform 1-distant trees are a special case of caterpillars, and uniform 2-distant trees are a special case of lobsters.

Progress, *continued*.

- It is known that all caterpillars are graceful (Rosa) and that all lobsters with a perfect matching are graceful (Morgan), yet it remains an open problem to determine if all lobsters are graceful.

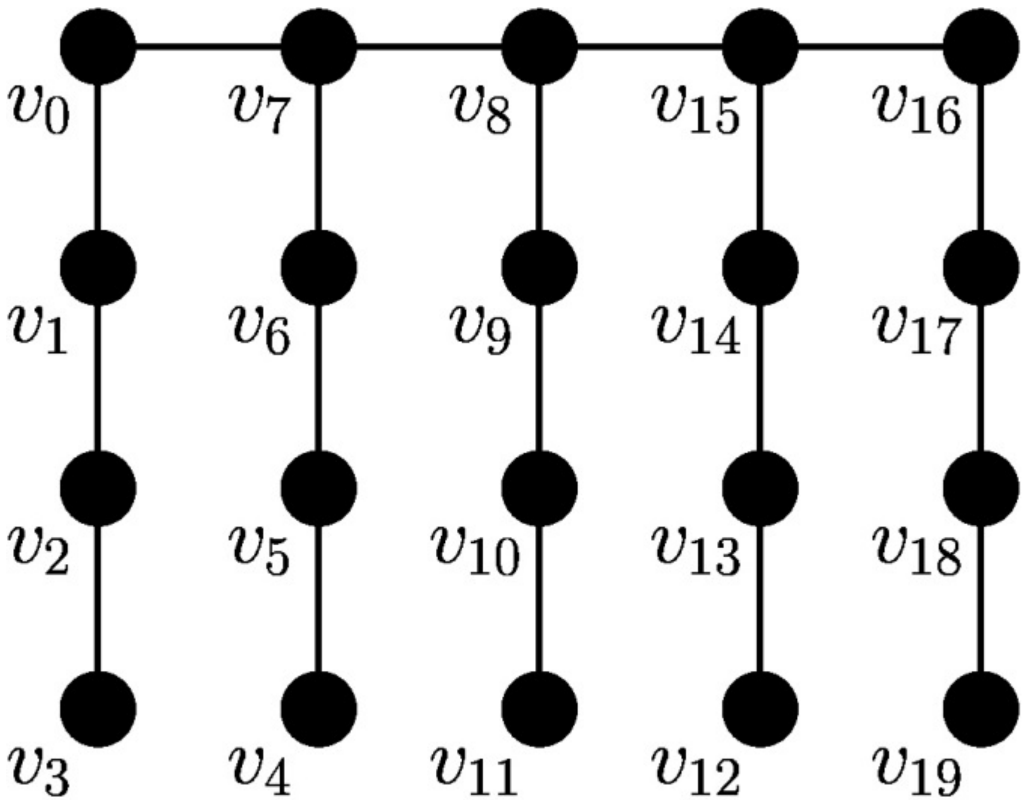
Results

Theorem 1.
Every uniform k-distant tree is graceful.

Let G be a uniform k-distant tree, where k is the number of edges in each tail, and let s denote the number of vertices in the spine. Let n be the number of vertices of G. Note that $n = s(k+1)$.

First, we name the vertices.

Naming the vertices

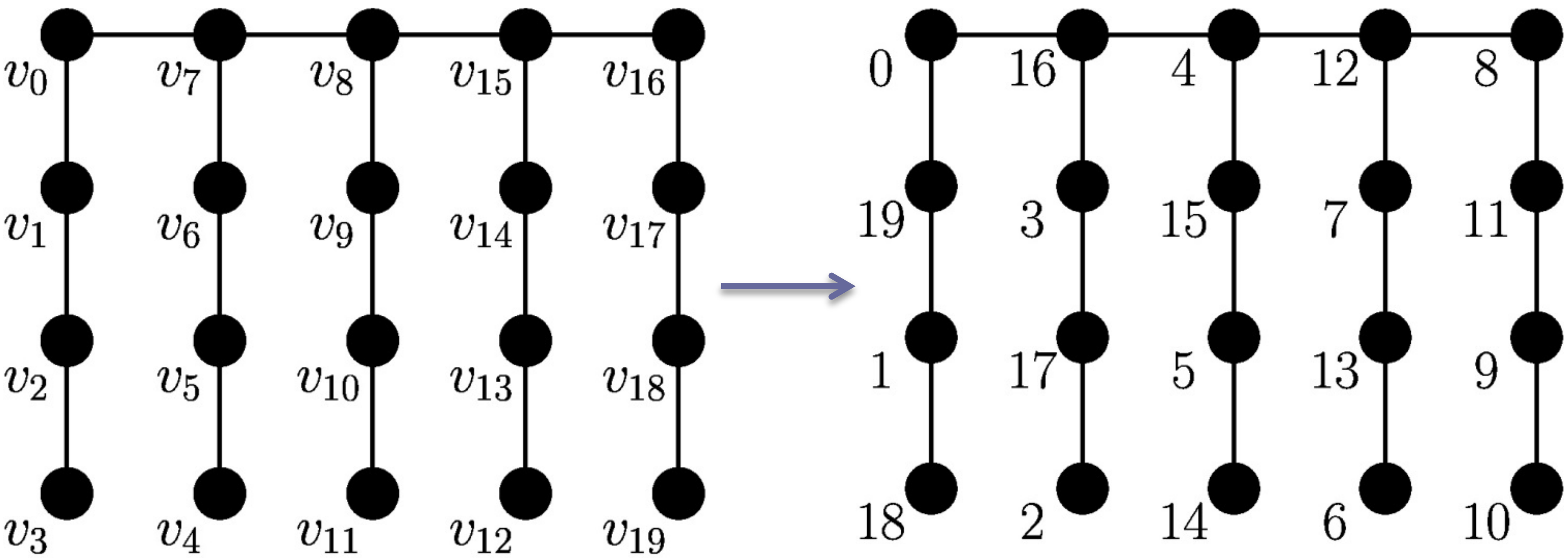


Labeling the vertices

Then, we assign labels to the vertices. To do so, we define a function *f* from the *n* vertices of the graph to the set {0,…,*n*-1}, where *n*-1 is the number of edges. We split this into two cases.

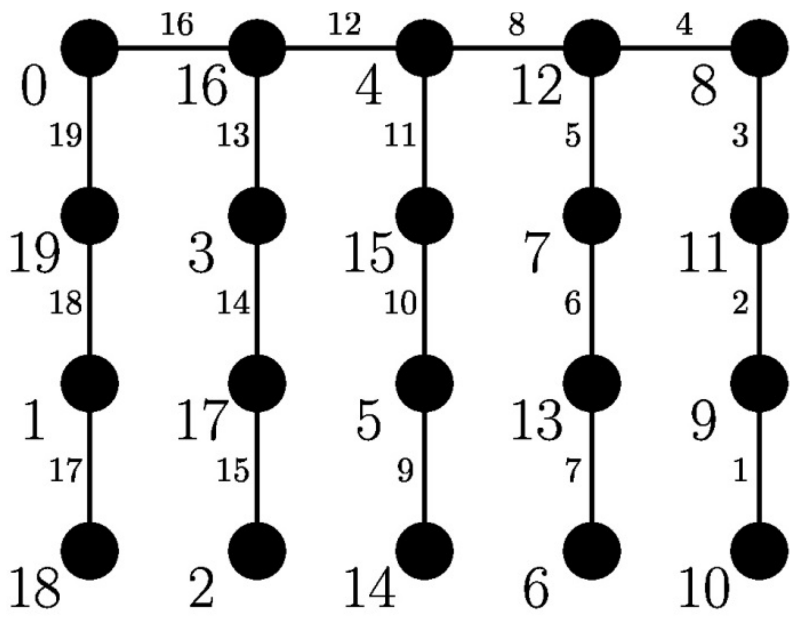
Case 1. n is even	Case 2. n is odd
$f(v_i) = \begin{cases} \frac{i}{2} & \text{if } i = 0, 2, 4, ..., n-2 \\ n - \frac{i+1}{2} & \text{if } i = 1, 3, 5, ..., n-1 \end{cases}$	$f(v_i) = \begin{cases} \frac{i}{2} & \text{if } i = 0, 2, 4, ..., n-1 \\ n - \frac{i+1}{2} & \text{if } i = 1, 3, 5, ..., n-2 \end{cases}$

Labeling the vertices, *continued*.



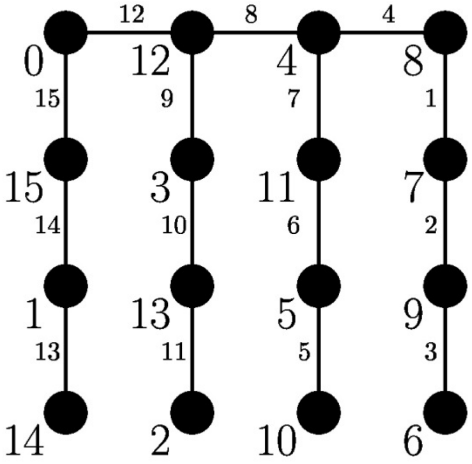
Is it graceful?

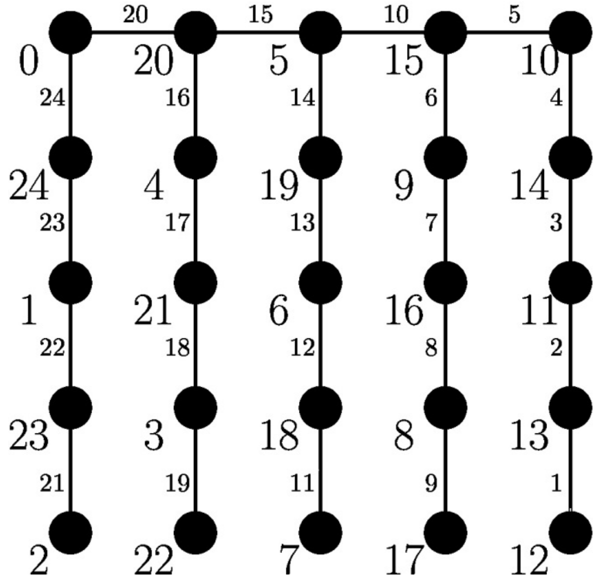
Recall that we have a graceful labeling if and only if each edge label given by the absolute value of the difference of the endpoint labels is unique.

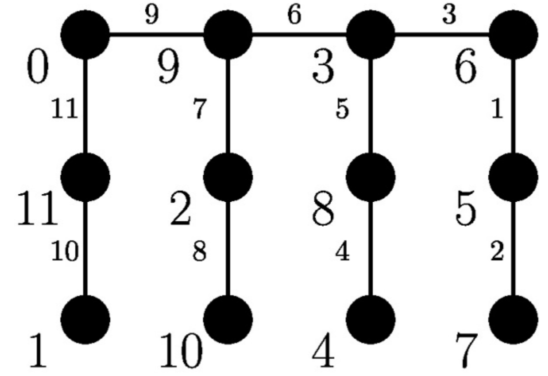


What if *k* and *s* aren't both odd?

k odd, s even	k even, s odd	k even, s even







Conclusions

- All uniform *k*-distant trees are graceful.
- It should be noted that the question about the gracefulness of uniform *k*-distant trees has been addressed elsewhere. After proving this result independently last summer, I recently discovered that M. Murugan has included this theorem in an article published in *Matematika*, also in Summer 2013.
- It is our hope that this method can be used to approach the question whether or not lobsters are graceful, inching us closer to proving the Ringel-Kotzig conjecture.

Conclusions, *continued.*

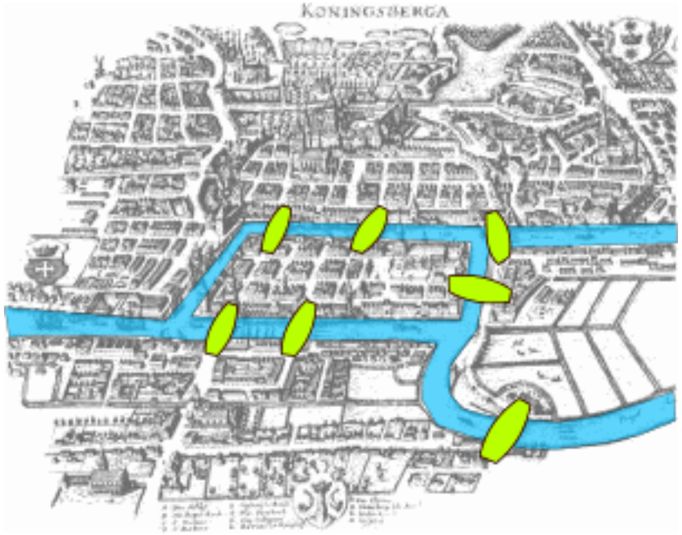
- A **harmonious labeling** of a graph G on n vertices is an injective function from the vertices of G to the set {0, … , $|E(G)|$} such that the induced edge labels given by $[f(u) + f(v)] (\text{mod } |E(G)|)$, for every uv in $E(G)$, are all distinct.
- Similar to graceful, but we add endpoint labels rather than subtract.
- For a tree, we may repeat one vertex label.

Conclusions, *continued*.

- A similar approach was used to show that all uniform k-distant even trees are harmonious (Abueida and Roberts). Murugan has recently shown that this is true for the odd case, as well.
- Gallian notes that whether or not lobsters are harmonious seems to have attracted no attention thus far. This is yet another open problem to which this method might be applied.

Where did this even come from?

Where did graph theory begin?



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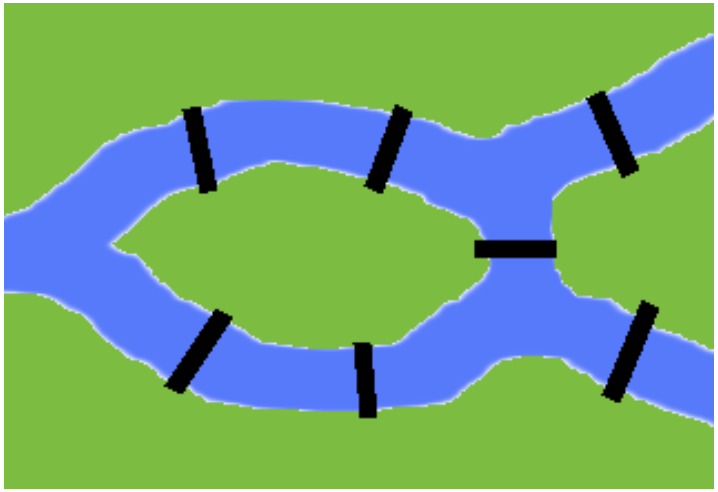

Image by MoRsE

Leonhard Euler used vertices and edges to model and solve this problem, laying the foundations for the study of graph theory.

Citations

- A. Abueida and Dan Roberts, *Uniform k-Distant Even Trees are Harmonious*, Utilitas Math. 78 (2009), 279-285.
- J.A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, DS6.
- M. Murugan, *(k,d)-Balanced of Uniform k-Distant Trees,* Matematika, 29 (2013), 65-71.
- M. Murugan, *Harmonious Properties of Uniform k-Distant Trees,* Chinese Journal of Mathematics, 2013.
- David Morgan, *All lobsters with perfect matchings are graceful*, Electron. Notes Discrete Math., 11, 2002.
- A. Rosa, *On certain valuations of the vertices of a graph, Theory of Graphs* (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.
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