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Multidecompositions of Complete Graphs into a Graph Pair of Order 6

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1 Abstract

We find both necessary condition and sufficient condition for (C_6, $\overline{C}_6$)multidecomposition of complete graph

2 Introduction

Here are basic definitions of graph theory. Let G and H be graphs. Denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The degree of a graph vertex V of a graph G is the number of graph edges which touch V. We say that G is connected if there is a path from any vertex in G to any other vertex in G. An isolated vertex is a vertex with degree of zero. The union of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the union of their vertex and edge sets: $G \cup H = (V_G \cup V_H,\, E_G \cup E_H)$

Let K_n denote the complete graph on n vertices. The complete graph on n vertices, denoted K_n, is the graph on n vertices such that every pair of distinct vertices has exactly one edge between them. Let C_m denote the cycle with m vertices. A cycle on n vertices containing a single cycle through all vertices. Given graph G on n vertices, define $\overline{G}$ as the graph with n vertices such that $E(\overline{G}) = E(K_n)/E(G)$ when considering G as a subgraph of $V_G = V_H$. In other words, $\overline{C}_m$ is the complement of $\overline{C}_m$ to K_n.

Given graphs G and H, a $G-decomposition$ of H is a set $\{G_1, G_2, ..., G_t\}$ of edge-disjoint subgraphs of H such that $\bigcup_{i=1}^{t} E(G_i) = E(H)$ and $G_i \cong G$ for every $i \in \{1, ..., t\}$. If a $G - decomposition$ of H exists, then we say that G *decomposes* H or H *decomposes into copies of* G.

A $(G, H) - multidecomposition$ of K_n is a set S = $\{S_1,\ S_2,\ S_3, ..., S_t\}$ of edge-disjoint subgraphs of K_n such that $\bigcup_{i=1}^{t} E(S_i) = E(K_n)$, $S_i \cong G$ or $S_i \cong H$ for every $i \in$ {1,...,t}. and at least one copy of G and one copy of H are included in S. Let G and H be edge-disjoint connected spanning subgraphs of K_n. We call (G, H) a *graph pair of order* n if $E(G) \cup E(H) = E(K_n)$. Multidecompositions of complete graphs into graph pairs of orders 4 and 5 have been studied, and the following results were obtained. Denote the graph consisting of two vertex-disjoint edges by $2K_4$.

Theorem 2.1 (Abueida and Daven [?]). *There is a (C_4, $2K_2$)-multidecomposition of K_m if and only if* $m \equiv 0, 1 \pmod 4$ *($m \geq 4, m \neq 5$)*

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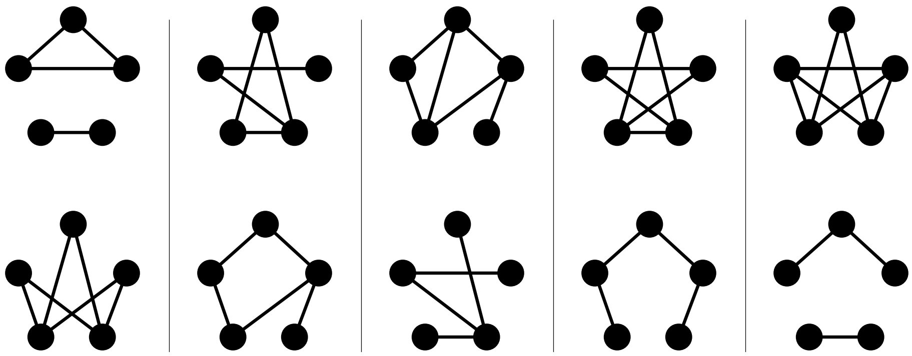

Figure 1: The graph pairs of order 5.

Theorem 2.2 (Abueida and Daven [?]). *There is a (G_i, H_i)-multidecomposition of K_m for $m \geq 5$ if and only if*

1. *when $i \in \{1,3,4\}$, $m \equiv 0,1 \pmod 4$ (except when $i = 1$ and $m = 8$);*
2. *when $i = 2, m \equiv 0,1 \pmod 5$;*
3. *when $i = 5, m \neq 6,7$;*

In this paper, we investigate multidecompositions of complete graphs of order 6. In particular, we find necessary and sufficient conditions for the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_n. Our main result is as follows.

Theorem 2.3. *The complete graph K_n admits a $(C_6, \overline{C}_6)$-multidecomposition of K_n if and only if $n \equiv 0,1,3,4 \pmod 6$ with $n \geq 6$, except $n \in \{7,9,10\}$, and possibly except $n = 19$.*

Let G and H be vertex-disjoint graphs.The *join of G and H*, denoted $G \vee H$, is defined to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u,v\} \colon u \in V(G), v \in V(H)\}$. We use the shorthand notation $\bigvee_{i=1}^{t} G_i$ to denote $G_1 \vee G_2 \vee \cdots \vee G_t$, and when $G_i \cong G$ for all $1 \leq i \leq t$ we write $\bigvee_{i=1}^{t} G$. For example, $K_{12} \cong \bigvee_{i=1}^{4} K_3$. Let $(a_0, a_1, ..., a_{n-1})$ denote the cycle on n vertices with vertex set $\{a_0, a_1, ..., a_{n-1}\}$ and edge set $\{\{a_i, a_{i+1}\} | i = 0, 1, ..., n-2\} \cup \{\{a_0, a_{n-1}\}\}$. Let [a,b,c;d,e,f] denote the graph with vertex set {a,b,c,d,e,f} and edge set

$$\{\{a,b\},\{b,c\},\{a,c\},\{d,e\},\{e,f\},\{d,f\},\{a,d\},\{b,e\},\{c,f\}\}.$$

Notice that [a,b,c;d,e,f] is isomorphic to $\overline{C}_6$.

Next, we introduce some results on graph decompositions that will help us prove our main result. Sotteau's theorem can be used to decompose bipartite graphs into even length cycles.

Theorem 2.4 (Sotteau [?]). *A C_{2k}-decomposition of $K_{m,n}$ exists if and only if $m \geq k$, $n \geq k$, m and n are both even, and $2k$ divides mn.*

Specifically, we use Sotteau's theorem to obtain C_6-decompositions of complete bipartite graphs.

Corollary 2.5 (Sotteau [?]). *A C_6-decomposition of $K_{m,n}$ exists if and only if $m \geq 3$, $n \geq 3$, m and n are both even, and 6 divides mn.*

A celebrated result in the field of graph decompositions is that the necessary conditions for a C_k-decomposition of K_n are also sufficient. Here we state the result only for $k = 6$.

Theorem 2.6 (Alspach et al. [?]). *Let n be a positive integer. A C_6-decomposition of K_n exists if and only if $n \equiv 1 \text{ or } 9 \pmod{12}$.*

The necessary and sufficient conditions for a $\overline{C}_6$-decomposition of K_n are also known, and stated in the following theorem.

Theorem 2.7 (Kang et al. [?]). *Let n be a positive integer. A $\overline{C}_6$-decomposition of K_n exists if and only if $n \equiv 1 \pmod{9}$.*

Now, we define a type of graph labeling which will help to construct certain graph decompositions. A *σ-labeling* of a graph G on n edges is a one-to-one function $f : V(G) \rightarrow \{0, ..., 2n\}$ such that the set of induced edge labels given by $|f(u) - f(v)|$, for every $\{u, v\} \in E(G)$, forms the set $\{1, 2, \ldots, n\}$. In 1967, Rosa introduced graph labelings as a means to find graph decompositions. The connection between σ-labelings and graph decompositions is apparent in the following theorem, which follows directly from results in [?].

Theorem 2.8. *(A.Rosa [?]) Let G be a graph on n edges. If G admits a σ-labeling, then a cyclic G-decomposition of K_{2n+1} exists.*

3 Main Result

Lemma 3.1. *Let $n \geq 2$ be an integer. If $n \equiv 2 \text{ or } 5 \pmod{6}$ then 3 does not divide $\binom{n}{2}$.*

Proof. Let $n \geq 2$ be an integer.
Case 1: $n \equiv 2 \pmod{6}$. Let $n = 6x + 2$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x + 2)(6x + 1)}{2} = 18x^2 + 9x + 1 \equiv 1 \pmod{3}.$$

Thus, 3 does not divide $\binom{n}{2}$.
Case 2: $n \equiv 2 \pmod{6}$. Let $n = 6x + 2$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x + 5)(6x + 4)}{2} = 18x^2 + 27x + 10 \equiv 1 \pmod{3}.$$

Thus, 3 does not divide $\binom{n}{2}$. □

Lemma 3.2. *Necessary conditions for the existence of a $(C_6, \overline{C}_6)$-multidecomposition of K_n are*

1. *$n \geq 6$, and*

2. *$n \equiv 0, 1, 3, 4 \pmod{6}$.*

Proof. Let $n \geq 0$ be an integer.
Case 1: $n \equiv 0 \pmod 6$. Let $n = 6x$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x)(6x-1)}{2} = 18x^2 - 3x \equiv 0 \pmod 3.$$

Thus, 3 divides $\binom{n}{2}$.
Case 2: $n \equiv 1 \pmod 6$. Let $n = 6x + 1$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x+1)(6x)}{2} = 18x^2 + 3x \equiv 0 \pmod 3.$$

Thus, 3 divides $\binom{n}{2}$.
Case 3: $n \equiv 3 \pmod 6$. Let $n = 6x + 3$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x+3)(6x+2)}{2} = 18x^2 + 15x + 3 \equiv 0 \pmod 3.$$

Thus, 3 divides $\binom{n}{2}$.
Case 4: $n \equiv 4 \pmod 6$. Let $n = 6x + 4$ for some positive integer x. Then we have

$$\binom{n}{2} = \frac{(6x+4)(6x+3)}{2} = 18x^2 + 21x + 6 \equiv 0 \pmod 3.$$

Thus, 3 divides $\binom{n}{2}$.

□

Lemma 3.3. *No $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_7 exists.*

Proof. Note that K_7 has 21 edges. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_7 to exist, there must exist positive integers x and y such that $21 = 6x + 9y$. The only possibility is $x = 2$ and $y = 1$. This implies that a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_7 must contain exactly one copy of $\overline{C}_6$.

Notice that the degree of every vertex in K_7 is 6. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_7 to exist, there must exist positive integers p and q such that $6 = 2p + 3q$. The only possibilities are $(p, q) \in \{(3, 0), (0, 2)\}$. Since a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_7 requires at least one copy of $\overline{C}_6$, there must exist at least one vertex in K_7 that is contained in 2 copies of $\overline{C}_6$. This is a contradiction to the fact that such a multidecomposition must contain exactly one copy of $\overline{C}_6$. □

Lemma 3.4. *No $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 exists.*

Proof. Note that K_9 has 36 edges. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 to exist, there must exist positive integers x and y such that $36 = 6x + 9y$. The only possibility is $x = 3$ and $y = 2$. This implies that a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 must contain exactly two copies of $\overline{C}_6$.

Notice that the degree of every vertex in K_9 is 8. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 to exist, there must exist positive integers p and q such that $8 = 2p + 3q$. The only possibilities are $(p,q) \in \{(4,0),(1,2)\}$. Since a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 requires exactly two copies of $\overline{C}_6$, so there must exist at least one vertex $a \in V(K_9)$ that is contained in exactly one copy of $\overline{C}_6$. However, this contradictions the fact that vertex a must be contained in either 0 or 2 copies of $\overline{C}_6$. □

Lemma 3.5. *No $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_{10} exists.*

Proof. Note that K_{10} has 45 edges. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 to exist, there must exist positive integers x and y such that $45 = 6x + 9y$. The only possibilities are $(x,y) \in \{(6,1),(3,3)\}$. This implies that a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_9 must contain at least one $\overline{C}_6$. However, if such a multidecomposition consisting of exactly one copy of $\overline{C}_6$ existed, then the vertices which are not included in this copy would have odd degrees remaining after the removal of the copy of $\overline{C}_6$. Thus, the case where $(x,y) = (6,1)$ is impossible.

Notice that the degree of every vertex in K_{10} is 9. Therefore, for a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_{10} to exist, there must exist positive integers p and q such that $9 = 2p + 3q$. The only possibilities are $(p,q) \in \{(3,1),(0,3)\}$.

Assume we have a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition, $\mathcal{G}$, of K_n. Let $A, B, C \in \mathcal{G}$ with $A \cong B \cong C \cong \overline{C}_6$. Let $X = V(A) \cap V(B)$. It must be the case that $|X| \geq 2$ since K_{10} has 10 vertices. It also must be the case that $|X| \leq 5$ since K_6 does not contain two copies of $\overline{C}_6$.

If $|X| \in \{2,3\}$, then $V(C) \cap (V(A) \triangle V(B)) \neq \emptyset$. This implies that there exists a vertex in $V(K_n)$ that is contained in exactly 1 copy of $\overline{C}_6$ in $\mathcal{G}$, which is a contradiction.

Now, we make the observations that any set of either 4 or 5 vertices in $\overline{C}_6$ must induce at least 3 or 6 edges, respectively. Furthermore, the vertices in X must necessarily be contained in $V(C)$ due to the degree constraints put in place by the existence of $\mathcal{G}$. If $|X| = 4$ or $|X| = 5$, then X must induce at least 9 or at least 18 edges, respectively. This is a contradiction. Thus, no such $\mathcal{G}$ exists.

□

Lemma 3.6. *If $n \equiv 0 \pmod 6$ with $n \geq 6$, then K_n admits a $(C_6,\overline{C}_6)$-multidecomposition.*

Proof. Let $n = 6x$ for some integer $x \geq 1$. Note that $K_{6x} \cong \bigvee_{i=1}^{x} K_6$. On each copy of K_6 place a $(C_6, \overline{C}_6)$-multidecomposition of K_6. The remaining edges form edge-disjoint copies of $K_{6,6}$, which admits a C_6-decomposition by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n. □

Lemma 3.7. *If $n \equiv 1 \pmod 6$ with $n \geq 13$, then K_n admits a $(C_6,\overline{C}_6)$ -multidecomposition except possibly when $n = 19$.*

Proof. First, we need two building blocks for our general constructions. Let $V(K_{13}) = \{1, 2, \ldots, 19\}$. The following is a $(C_6, \overline{\mathrm{C}}_6)$-multidecomposition of K_{13}.

$$\{[1,2,3;7,0,8],[1,4,5;9,12,10],[3,4,6;7,11,10],[2,5,6;8,12,11]\}$$
$$\cup\,\{[13,1,6,8,5,11],[13,2,3,5,9,4,10],[13,3,5,9,4,10],$$
$$[13,7,12,3,9,6],[13,8,10,2,7,5],[13,9,11,1,8,4]\}$$

The second building block is a $\overline{C}_6$-decomposition of K_{19}. It is known (see Theorem **??**) that such a decomposition exists. However, here we provide a cyclic $\overline{C}_6$-decomposition of K_{19}.

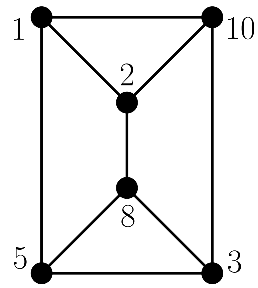

Figure 2: A σ-labeling of $\overline{C}_6$.

The labeling provided in Figure **??** is a σ-labeling of $\overline{C}_6$. Thus, by Theorem **??** there exists a cyclic $\overline{C}_6$-decomposition of K_{19}.
Let $n = 6x + 1$ for some integer $x \geq 2$.
Case 1: $x = 2k$ for some $k \geq 1$. Notice that $K_{12x+1} \cong K_1 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. On each of the k copies of K_{13} formed by $K_1 \vee K_{12}$, we place a $(C_6, \overline{C}_6)$-multidecomposition of K_{13}. The remaining edges form edge-disjoint copies of $K_{12,12}$. On each of the copies of $K_{12,12}$ we place a C_6-decomposition of $K_{12,12}$, which is known to exist by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n.
Case 2: $x = 2k + 1$ for some $k \geq 2$. Notice that $K_{12x+7} \cong K_1 \vee K_6 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. On the copy of K_{19} formed by $K_1 \vee K_6 \vee K_{12}$, we place a cyclic $\overline{C}_6$-decomposition of K_{19}. On the remaining $k-1$ copies of K_{13} formed by $K_1 \vee K_{12}$, we place a $(C_6, \overline{C}_6)$-multidecomposition of K_{13}. The remaining edges form edge-disjoint copies of either $K_{6,12}$ or $K_{12,12}$. Both of these graphs admit C_6-decompositions by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n.
We note that this proof does not provide a $(C_6, \overline{C}_6)$-multidecomposition of K_{19}. It is not known whether such a multidecomposition exists. □

Lemma 3.8. *K_{15} admits a (C_6,$\overline{C}_6$) -multidecomposition.*

Proof. Let $V(K_{15}) = A \cup B \cup C$ where $A = \{A_i : 0 \leq i \leq 4\}$, $B = \{B_i : 0 \leq i \leq 4\}$, and $C = \{C_i : 0 \leq i \leq 4\}$. Also let

$$\begin{aligned} E(K_{15}) = \{&\{X_i, X_j\} : X \in \{A, B, C\} \text{ and } 0 \leq i < j \leq 4\} \\ &\cup \{\{X_i, Y_j\} : X, Y \in \{A, B, C\}, X \neq Y, \text{ and } 0 \leq i \leq j \leq 4\}. \end{aligned}$$

Notice that the edges of K_{15} are of one of two types, either both endpoints are in one of A, B, or C, or the endpoints come from different sets. We wish to define the *difference* of an edge, and this definition depends on which type of edge is under consideration. Consider an edge of the form $e = \{X_i, X_j\}$ where $X \in \{A, B, C\}$, and let $d = \min\{|i-j|, 5-|i-j|\}$. We define the difference of e to be d_X, and we refer to differences of this type as *pure differences*. Now, consider an edge of the form $e = \{X_i, Y_j\}$ where $X \neq Y$, and let $d = |j - i|$. Assume that (X, Y) is of one of the forms $(A, B), (B, C)$, or (A, C). We define the difference of e to be d_{XY}, and we refer to differences of this type as *mixed differences*. Thus, K_{15} consists of the set of pure differences $\{1_X, 2_X : X \in \{A, B, C\}\}$ and the set of mixed differences $\{i_{XY} : 0 \le i \le 4 \text{ and } XY \in \{AB, BC, AC\}\}$.

If G is a subgraph of K_{15}, then applying the permutation $i \mapsto i+1 \pmod 5$ to the subscripts of the vertices of G produces a different subgraph G' which is isomorphic to G and has the same differences as G. Therefore, to obtain a $(C_6, \overline{C}_6)$-multidecomposition of K_{15} it suffices to specify a set of edge-disjoint subgraphs in K_{15} which are isomorphic to either C_6 or $\overline{C}_6$, contain at least one of each, and partition the set of differences of K_{15}. Since K_{15} has 21 differences, we are lead to produce 2 edge-disjoint copies of C_6 and 1 copy of $\overline{C}_6$. The copies are as follows.

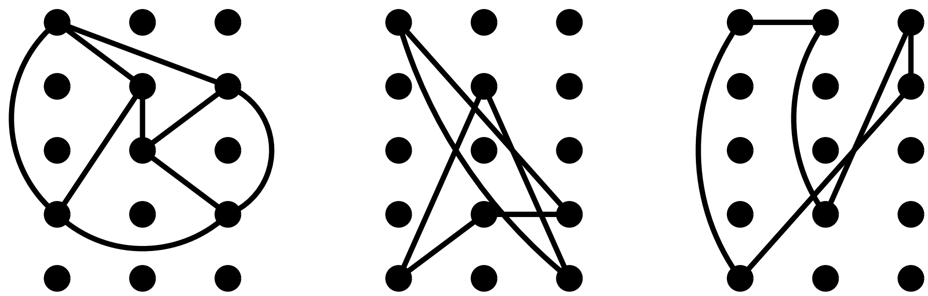

Figure 3: A labeling of $(C_6, \overline{C}_6)$-multidecomposition of K_{15}

The set of subgraphs obtained by applying the permutation $i \mapsto i+1 \pmod 5$ 4 times to each subgraph above will produce the desired $(C_6, \overline{C}_6)$-multidecomposition of K_{15}. □

Lemma 3.9. *If $n \equiv 3 \pmod 6$ with $n \ge 15$, then K_n admits a $(C_6,\overline{C}_6)$ -multidecomposition.*

Proof. **Case 1**: $x = 2k$ for some $k \ge 1$.

Notice that $K_{6x+3} \cong K_1 \vee K_{14} \vee \left(\bigvee_{i=1}^{k-1} K_{12}\right)$. On each of the $k-1$ copies of K_{13} formed by $K_1 \vee K_{12}$, we place a $(C_6, \overline{C}_6)$-multidecomposition of K_{13}. On the copy of K_{15} formed by $K_1 \vee K_{14}$ we place a $(C_6, \overline{C}_6)$-multidecomposition of K_{15} constructed in Lemma **??**. The remaining edges form edge-disjoint copies of $K_{12,12}$ and a $K_{12,14}$. Both of these complete bipartite graphs admit C_6-decompositions by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n.

Case 1: $x = 2k+1$ for some $k \ge 1$. Notice that $K_{6x+3} \cong K_1 \vee K_8 \vee \left(\bigvee_{i=1}^{k} K_{12}\right)$. On each of the k copies of K_{13} formed by $K_1 \vee K_{12}$, we place a $(C_6, \overline{C}_6)$-multidecomposition of K_{13}. By Theorem **??**, K_9 admits a C_6-decomposition. The remaining edges form edge-disjoint copies of $K_{8,12}$ and a $K_{12,12}$. Both of these complete bipartite graphs admit C_6-decompositions by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n.

We note that this proof does not provide a $(C_6, \overline{C}_6)$-multidecomposition of K_{19}. It is not known whether such a multidecomposition exists. □

Lemma 3.10. *If $n \equiv 4 \pmod 6$ with $n \ge 16$, then K_n admits a $(C_6,\overline{C}_6)$ -multidecomposition.*

Proof. First, we need two building blocks for our general constructions.

Let $V(K_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The following is a $(C_6, \overline{C}_6)$-multidecomposition of K_{10}.

$$\{[1, 2, 10; 6, 5, 7], [2, 3, 4; 8, 9, 10], [3, 7, 8; 5, 9, 4], [2, 6, 9; 7, 4, 1], [3, 6, 10; 1, 8, 5]\}.$$

Let $n = 6x + 4$ where $x \geq 2$ is an integer. Note that $K_{6x+4} \cong K_{10} \vee \left(\bigvee_{i=1}^{x-1} K_6 \right)$. On the copy of K_{10} place a $\overline{C}_6$-decomposition of K_{10} found above. On each copy of K_6 place a $(C_6,\overline{C}_6)$ -multidecomposition. The remaining edges form edge-disjoint copies of $K_{6,6}$ and $K_{6,10}$. Both of these complete bipartite graphs admit C_6-decompositions by Corollary **??**. Thus, we obtain the desired $(C_6, \overline{C}_6)$-multidecomposition of K_n. □

4 Conclusion

According to the necessary and sufficient conditions, we are able to conclude that K_n admits a $(C_6, \overline{C}_6)$-multidecomposition if and only if $n \equiv 0, 1, 3, 4 \pmod 6$ and $n \geq 6$ except for $n \in \{7, 9, 10\}$, and possibly except for $n = 19$. We will continue to study if K_{19} allows a $(C_6,\overline{C}_6)$ multidecomposition in the future.

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