Multidecompositions of complete graphs into a graph pair of order 6

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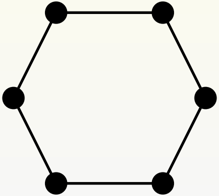
Definitions

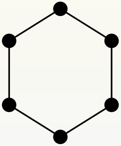
A n-cycle, denoted C_n, is a connected 2-regular graph on n vertices.



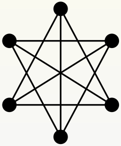
6-cycle

Definitions

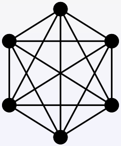
Given a graph G on n vertices, the complement of G, denoted $\overline{G}$, is the graph on the same vertex set with edge set $E(K_n) \setminus E(G)$.



C_6



$\overline{C}_6$



K_6

Definitions

Let G and H be graphs. A G-decomposition of H is a partition of the edges of H into copies of G.

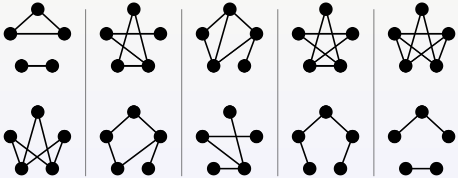
Let G and H be graphs. A (G, H)-multidecomposition of order n is a partition of the edges of K_n into copies of G and H with at least one copy of G and at least one copy of H.

Definitions

Let G and H be graphs. We call (G, H) a graph pair of order n if all of the following hold.

1. Both G and H have n vertices, none of which are isolated,
2. $G \ncong H$, and
3. $G \cup H = K_n$.

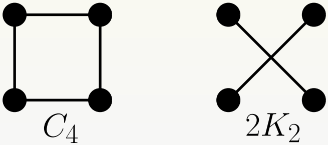
Figure: The graph pairs of order 5.



The graph pair of order 4

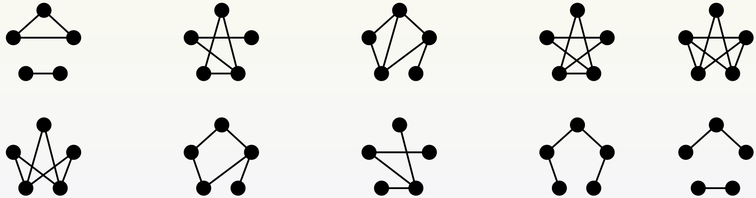
Theorem (Abueida and Daven, 2003)

There is a $(C_4, 2K_2)$-multidecomposition of order n if and only if $n \equiv 0, 1 \pmod 4$ where $n \geq 4$ and $n \neq 5$.



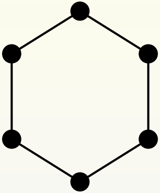
Graph pairs of order 5

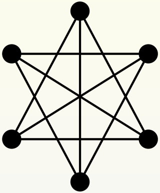
Let (G, H) be a graph pair of order 5. The necessary and sufficient conditions for (G, H)-multidecompositions of order n are as shown below (Abueida and Daven, 2003).
Assume that $n \geq 5$.

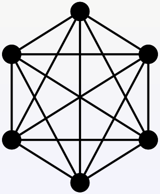


$n \equiv 0, 1 \pmod 4$, $n \neq 8$ | $n \equiv 0, 1 \pmod 5$ | $n \equiv 0, 1 \pmod 4$ | $n \equiv 0, 1 \pmod 4$ | $n \geq 8$

The graph pair $(C_6, \overline{C}_6)$







C_6-decompositions

Theorem (Rosa, 1966)

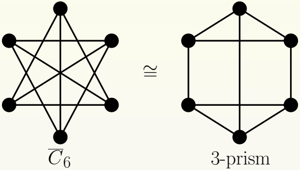
A C_6-decomposition of K_n exists if and only if $n \equiv 1, 9 \pmod{12}$.

Theorem (Sotteau, 1981)

A C_6-decomposition of $K_{m,n}$ exists if and only if

1. *m and n are both even,*
2. *$m, n \geq 3$, and*
3. *6 divides mn.*

$\overline{C}_6$-decompositions of K_n



Theorem (Kang et al., 2008)

A $\overline{C}_6$-decomposition of K_n exists if and only if $n \equiv 1 \pmod 9$.

Necessary conditions for a $(C_6, \overline{C}_6)$-multidecomposition of order n

(Order) $n \geq 6$,

(Size) $\dfrac{n(n-1)}{2} = 6x + 9y$ for some $x, y \geq 1$, and

(Degree) $n - 1 = 2p + 3q$ for some $p, q \geq 0$.

Therefore $n \equiv 0, 1, 3, 4 \pmod 6$

Case: $n = 6x + 1$

A $(C_6, \overline{C}_6)$-multidecomposition of order 7 does not exist.

$$\binom{7}{2} = 21 = 6x + 9y \Rightarrow x = 2 \text{ and } y = 1$$
$$6 = 2p + 3q \Rightarrow (p = 0 \text{ and } q = 2) \text{ or } (p = 3 \text{ and } q = 0)$$

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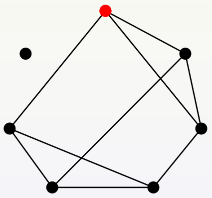
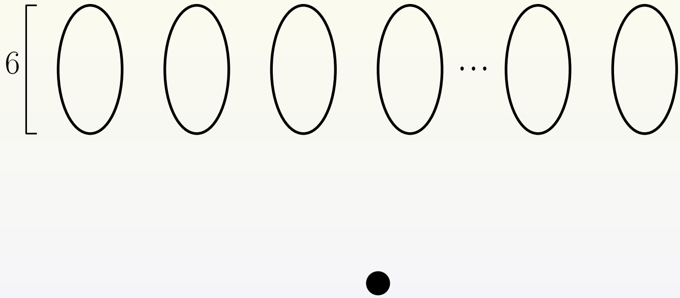


Case: $n = 6x + 1$ where x is even.



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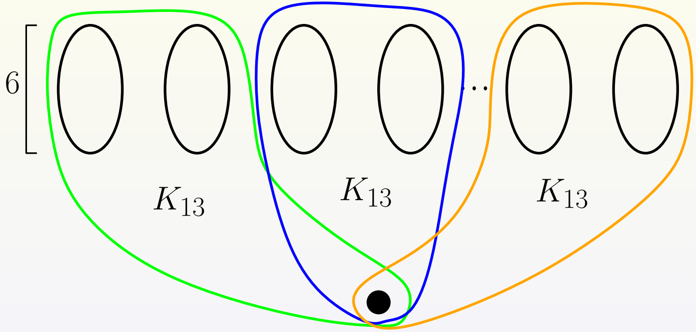

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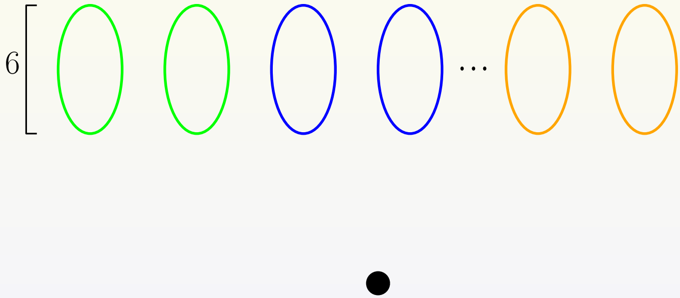

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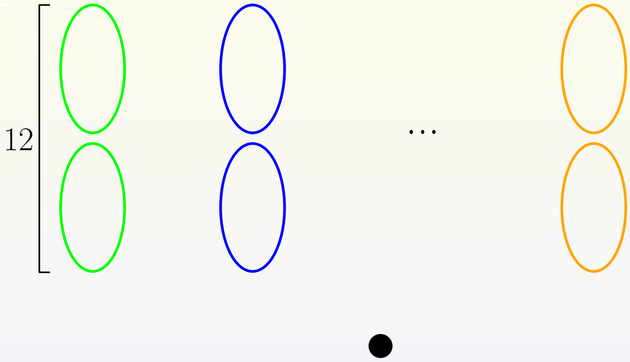

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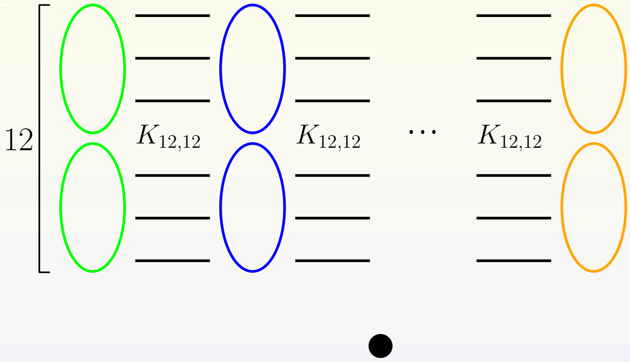
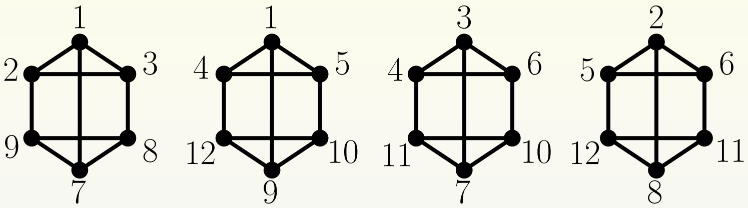


A $(C_6, \overline{C}_6)$-multidecomposition of order 13

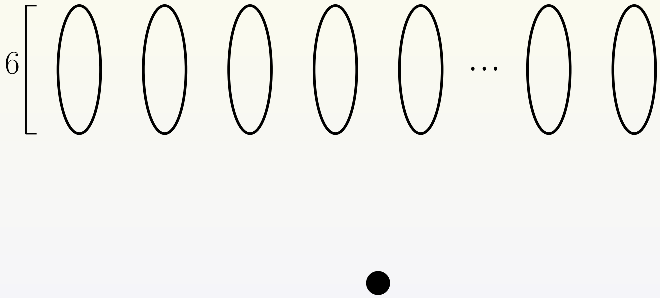


$$[13, 1, 6, 8, 5, 11] \quad [13, 2, 4, 7, 6, 12] \quad [13, 3, 5, 9, 4, 10]$$

$$[13, 7, 12, 3, 9, 6] \quad [13, 8, 10, 2, 7, 5] \quad [13, 9, 11, 1, 8, 4]$$

$$[1, 10, 3, 11, 2, 12]$$

Case: $n = 6x + 1$ where x is odd.



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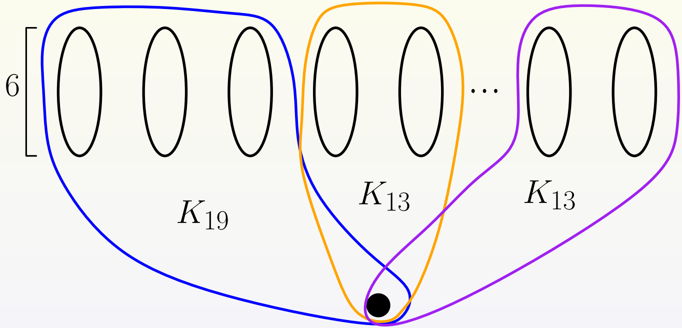

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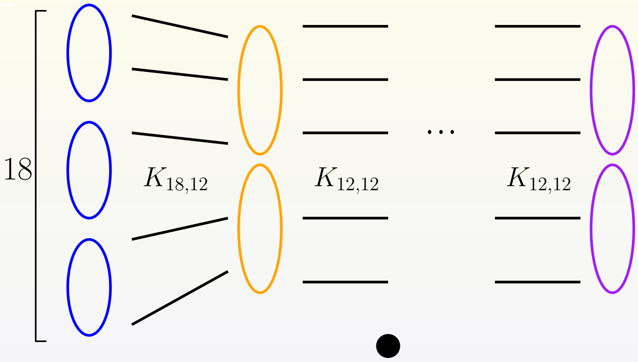
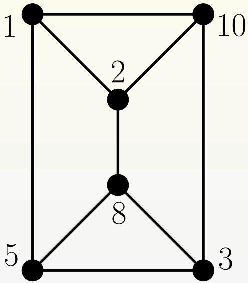


A cyclic $\overline{C}_6$-decomposition of K_{19}



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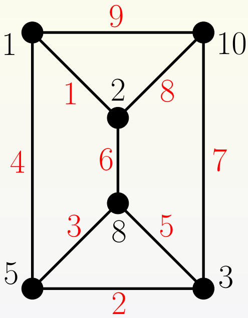

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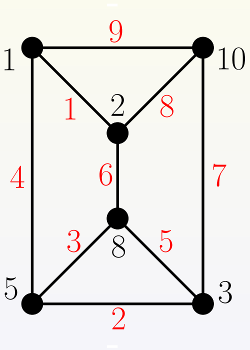

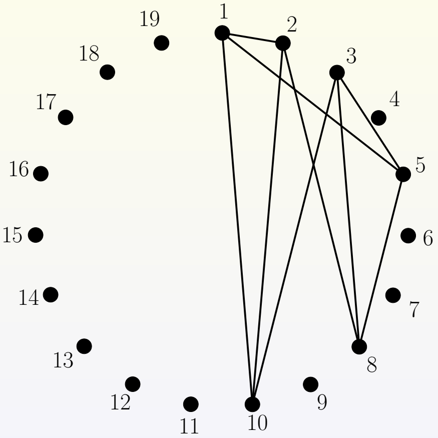

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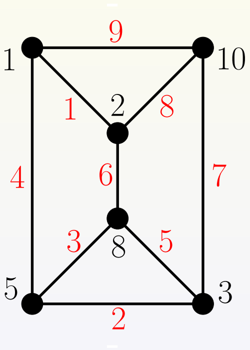

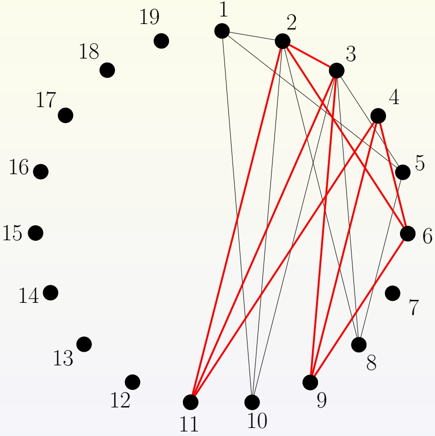

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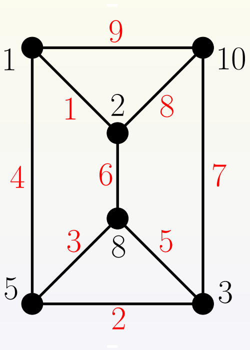

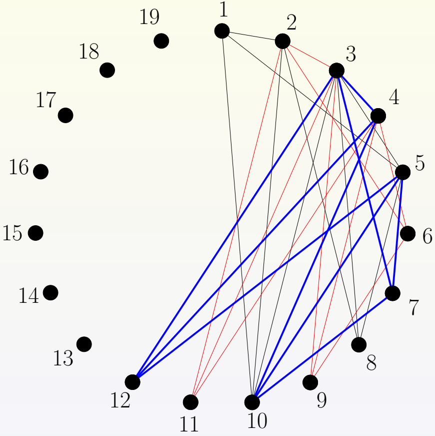

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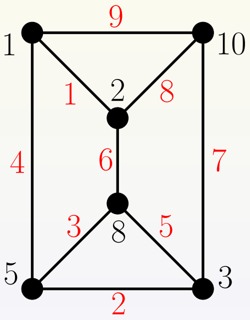

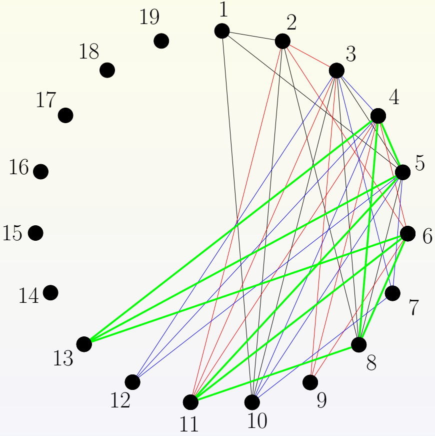

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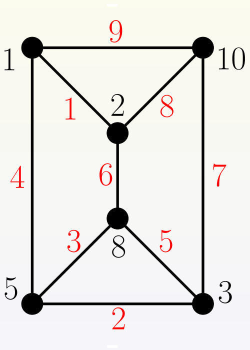

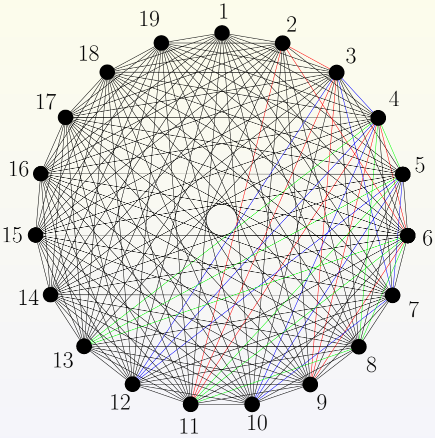

Theorem

A $(C_6, \overline{C}_6)$-multidecomposition of order n exists if and only if $n \equiv 0, 1, 3, 4 \pmod 6$, unless $n \in \{7, 9, 10\}$ and possibly $n = 19$.

More lines of inquiry.

- Multidecompositions into other graph pairs of order 6.

- $(C_6, \overline{C}_6)$-multidecompositions of order n with prescribed numbers of C_6 and $\overline{C}_6$.

Thank You

References

- A. Abueida and M. Daven, Multidesigns for Graph-Pairs of Order 4 and 5, *Graphs and Combinatorics* (2003) 19, 433–447.
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