



1996

# Macroelements and Orthogonal Multiresolutional Analysis

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## Recommended Citation

Corbett '96, Jonathan M., "Macroelements and Orthogonal Multiresolutional Analysis" (1996). *Honors Projects*. Paper 10.  
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## MACROELEMENTS AND ORTHOGONAL MULTIREOLUTIONAL ANALYSIS.

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Orthogonal multiresolutional wavelet analysis in a two dimension setting furnishes a basis for wavelet analysis. Bernstein-Bezier polynomials over simplexes provide elegant expressions of the necessary and sufficient conditions for a shift invariant space generating an orthogonal multiresolution analysis. In order to give the expressions, a formula of the inner product of two Bernstein-Bezier polynomials over a simplex has been derived:

$$\langle P_n, Q_n \rangle_{\Delta_s} = \iint_{\Delta_s} P_n(x) Q_n(x) dx = S! V_{\Delta_s} \sum_{i,j} a_i b_j (n!)^2 / (2n+s)! \prod_{k=1}^s \binom{i_k + j_k}{i_k}$$

where  $V_{\Delta_s}$  is the volume of the  $s$ -dimensional simplex,  $\mathbf{i} = i_1 + i_2 + \dots + i_s$ ,  $\mathbf{j} = j_1 + j_2 + \dots + j_s$ , and  $a_i$  and  $b_j$  are respective Bernstein-Bezier coefficients of  $P_n(x)$  and  $Q_n(x)$ . We also give the needed expression by using the formula above.

### 1. Introduction

#### 1.1) Definition of $L^2(\mathbf{R})$ space.

$L^2(\mathbf{R})$  space is defined as a collection of square integrable functions, i.e.  $\{f \mid \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty\}$ .  $L^2(\mathbf{R})$  is also an inner product space in which an inner product,  $\langle f, g \rangle$ ,  $f, g \in L^2(\mathbf{R})$ , is defined.

Here the inner product can be understood as an extension of the dot product. The inner product is thus an infinite summation of dot products in essence. So,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

whereas the dot product is denoted as:

$$\mathbf{v} \bullet \mathbf{g} = \sum_{i=0}^n v_i g_i.$$

for  $\mathbf{v} = (v_0, v_1, \dots, v_n)$  and  $\mathbf{g} = (g_0, g_1, \dots, g_n)$ .

So,  $L^2(\mathbf{R})$  space is defined as

$$L^2(\mathbf{R}) = \{f \mid \|f\|^2 = \langle f, f \rangle < \infty\}.$$

$L^2(\mathbf{R})$  is also a normed linear space, in which the norm  $\|f\| = [\langle f, f \rangle]^{1/2}$  is defined.

### 1.15) Additional Definitions in $L^2(\mathbf{R})$ .

It is helpful to the understanding of the following sections to define a few key concepts.

The first necessary concept is orthogonality as understood in  $L^2(\mathbf{R})$ . Two functions are orthogonal if their inner product is equal to zero, i.e.  $\langle f_1, f_2 \rangle = 0$ . Orthonormal sets are sets of normalized orthogonal elements. Therefore an orthonormal set, which has an infinite number of elements, can be denoted as:

$$\{f_1, f_2, f_3, \dots, f_n, \dots\} \text{ such that } \langle f_i, f_j \rangle = \delta_{ij}$$

where  $\delta_{ij}$  is the delta function defined as  $\delta_{ij} = 0$  when  $i \neq j$  and  $\delta_{ij} = 1$  when  $i = j$ .

The set  $\{f_1, f_2, \dots, f_n, \dots\}$  in  $L^2(\mathbf{R})$  is a complete orthogonal (or orthonormal) set if and only if  $\forall f \in L^2(\mathbf{R}), \langle f, f_n \rangle = 0, n = 1, 2, \dots$  implies  $f = 0$  almost everywhere. Loosely understood every  $f_i, f_j \in L^2(\mathbf{R})$  such that  $\langle f_i, f_j \rangle = 0$ , where  $f_i, f_j \neq 0$  must be a member of  $\{f_1, f_2, \dots, f_n, \dots\}$  for the mentioned set to be complete. By almost everywhere we mean for all  $x \in \mathbf{R}$  in the sense of some measurement  $f(x) = 0$ . If  $x$  has no measurement then  $f(x)$  is not necessarily 0.

We define the term dense in  $L^2(\mathbf{R})$  when referring to subspaces on  $L^2(\mathbf{R})$ . We say that a subspace is dense in  $L^2(\mathbf{R})$  if  $L^2(\mathbf{R})$  is contained in the closure of the subspace.

### 1.2) Approximation of $L(\mathbf{R})^2$ by use of orthonormal sets.

It can be shown that an orthonormal set,  $\{f_1, f_2, \dots, f_n, \dots\}$ , can be used to approximate functions in  $L(\mathbf{R})^2$  if it is complete. First we begin with the assumption that a function  $f \in L^2(\mathbf{R})$  can be expanded as:

$$f = \sum_{n=1}^{\infty} c_n f_n, \text{ where } \{f_1, f_2, \dots, f_n, \dots\} \text{ is orthonormal.}$$

Then  $\langle f, f_n \rangle = \langle \sum_k c_k f_k, f_n \rangle = c_n$ .

We must show that  $f = \sum_{n=1}^{\infty} c_n f_n$  is true. To do so we must show that the sequence of partial sums of series converges, and that it converges to  $f$ . We first show that the  $\sum_{n=1}^{\infty} c_n f_n$  is

convergent.

Define  $S_n = \sum_{k=1}^n c_k f_k$ . If the limit of  $S_n$  exists as  $n \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} c_n f_n = \lim_{n \rightarrow \infty} S_n$ . To show that  $\lim S_n$  does exist, we consider Bessel's inequality, which states  $\sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2$ . In fact we consider the  $L^2$  norm of  $f - S_n$ . (Note -- for a further understanding of Bessel's inequality, see reference)

(consider all  $\Sigma$  to run from 1 to  $\infty$  unless otherwise noted in paper)

$$\begin{aligned}
 \|f - S_n\|^2 &= \langle f - S_n, f - S_n \rangle \\
 &= \langle f - \sum c_j f_j, f - \sum c_k f_k \rangle \\
 &= \int_{-\infty}^{\infty} (f - \sum c_j f_j)(f - \sum c_k f_k) dx \\
 &= \int_{-\infty}^{\infty} (f^2 - f \sum c_k f_k - f \sum c_j f_j + \sum c_k f_k \sum c_j f_j) dx \\
 &= \langle f, f \rangle - \langle f, \sum c_k f_k \rangle - \langle f, \sum c_j f_j \rangle + \langle \sum c_j f_j, \sum c_k f_k \rangle \\
 &= \langle f, f \rangle - \sum c_k \langle f, f_k \rangle - \sum c_j \langle f, f_j \rangle + \sum \sum c_j c_k \langle f_j, f_k \rangle \\
 &= \langle f, f \rangle - \sum |c_k|^2 - \sum |c_k|^2 + \sum |c_k|^2 \\
 &= \|f\|^2 - \sum |c_k|^2.
 \end{aligned}$$

Since  $\|f - S_n\|^2 \geq 0$ , we obtain Bessel's inequality. The Bessel's inequality implies that  $\sum |c_n|^2$  is convergent, because  $\|f\|^2 < \infty$ . Then the sequence of partial sums  $S_n' := \sum_{k=1}^n |c_k|^2 > 0$  is increasing and bounded by  $\|f\|^2$ . It follows that  $\{S_n'\}$  is a Cauchy sequence.

To show that the sum  $\sum c_n f_n$  is convergent we will show that  $\{\sum_{k=0}^n c_k f_k\}$  is a Cauchy sequence. We have shown that for every  $\epsilon$ , there exists  $K$  such that for any  $n, m > K$ :  $|S_m - S_n| < \epsilon$ .

So,

$$\begin{aligned}
 |S_m - S_n| &= \left| \sum_{k=n+1}^m c_k f_k \right|^2 \\
 &= \langle \sum_{j=n+1}^m c_j f_j, \sum_{j=n+1}^m c_j f_j \rangle
 \end{aligned}$$

$$\begin{aligned}
&= \langle \sum_{j=n+1}^m c_j f_j, \sum_{k=n+1}^m c_k f_k \rangle \\
&= \sum_{j=n+1}^m c_j (\sum_{k=n+1}^m c_k \langle f_j, f_k \rangle) \\
&= \sum_{j=n+1}^m |c_j|^2 < \epsilon.
\end{aligned}$$

Therefore  $\{S_n\}$  is a Cauchy sequence, so  $\sum c_n f_n$  converges.

To show that  $\sum c_n f_n$  converges to  $f$  we must have the orthonormal set  $\{f_1, f_2, \dots, f_n, \dots\}$  to be complete in  $L^2(\mathbf{R})$  as previously defined. Because  $\sum c_n f_n$  converges, let  $g = \sum c_n f_n$ , where  $c_n = \langle f, f_n \rangle$  for each  $n$ . So,

$$\begin{aligned}
\langle f-g, f_n \rangle &= \langle f - \sum c_k f_k, f_n \rangle \\
&= \langle f, f_n \rangle - \langle \sum c_k f_k, f_n \rangle \\
&= c_n - \sum c_n \langle f_k, f_n \rangle \\
&= c_n - c_n \\
&= 0.
\end{aligned}$$

Since  $f-g = 0$  almost everywhere,  $f=g$ . Therefore  $\sum_{n=1}^{\infty} c_n f_n$  converges to  $f$  when  $\{f_1, f_2, \dots, f_n, \dots\}$  is a complete orthonormal sequence.

### 1.3) Wavelets

A wavelet is defined as a single function with the property that the set of its dilations and translations,  $\psi_{mn} = \{2^{m/2} \psi(2^m t - n)\}$ , give a complete orthogonal (not necessarily but possibly orthonormal) basis of  $L^2(\mathbf{R})$ .

### 1.4) Examples of complete orthogonal bases of $L^2$

#### A) Trigonometric complete orthogonal system in $L^2(-\pi, \pi)$ .

$L^2(-\pi, \pi)$  has a complete orthogonal basis  $\{f_1, f_2, \dots, f_n, \dots\}$ , where  $f_1 = 1/2$ ,  $f_2 = \sin x$ ,  $f_3 = \cos x$ ,  $f_4 = \sin(2x)$ ,  $f_5 = \cos(2x)$ , .... The inner product  $\langle f_n, f_m \rangle$  is defined by  $\int_{-\pi}^{\pi} f_n f_m dx = 0$  when  $n \neq m$

and  $\pi$  when  $n=m$ . We can show  $\{f_1, f_2, \dots, f_n, \dots\}$  to be complete on  $L^2(-\pi, \pi)$ . For any  $f \in L^2(-\pi, \pi)$  we can find a sum  $\sum c_n f_n = \sum \langle f, f_n \rangle f_n$  that converges to  $f$ .

## B) Wavelet Function

The Haar wavelet is denoted by  $\varphi(t)$  and is defined as:

$$\varphi(t) = X_{[0,1]}(t),$$

where  $X_{[0,1]}$  is the characteristic function defined as 1 on  $t = 0$  to 1 and 0 elsewhere.

The collection of its translations defined by  $\varphi(t-n)$ ,  $n \in \mathbf{N}$ , is an orthogonal system due to no overlap in translation; i.e.  $\langle \varphi(t), \varphi(t-n) \rangle = 0$  when  $n \neq 0$ . However, this system is not complete because  $\varphi(t-n)$  is not dense in  $L^2(\mathbf{R})$ .

We use the function  $\varphi_{mn}(t) := 2^{m/2} \varphi(2^m t - n)$ . Then  $\varphi_{mn}(t) = 1$  when  $2^{-m}n \leq t < (n+1)2^{-m}$  and  $\varphi_{mn} = 0$  elsewhere. Since any function in  $L^2(\mathbf{R})$  can be approximated by piecewise constant functions that may have the jump at binary rational numbers,  $\{\varphi_{mn}\}$  is a complete system in  $L^2(\mathbf{R})$ . However  $\{\varphi_{mn}\}$  is not an orthogonal set.

To construct a complete and orthogonal system in  $L^2(\mathbf{R})$ , we define a new function:  $\psi = \varphi(2t) - \varphi(2t-1)$ . The set  $\{\psi_{mn} = 2^{m/2} \psi(2^m t - n)\}$  is a complete orthonormal system in  $L^2(\mathbf{R})$ . The system is complete because function in  $L^2(\mathbf{R})$  can be expanded in  $\varphi_{mn}(t)$ . The system is orthonormal, which is confirmed by observation of the inner product  $\langle \psi_{mn}, \psi_{lk} \rangle$  defined on the system.

$$\begin{aligned} \langle \psi_{mn}, \psi_{lk} \rangle &= \int_{-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) 2^{l/2} \psi(2^l t - k) dt \\ &= 2^{(m+l)/2} \int_{-\infty}^{\infty} \psi(2^m t - n) \psi(2^l t - k) dt \\ &= 2^{(m+l)/2} \int_{-\infty}^{\infty} \psi(t') \psi(2^{l-m}(t'+n) - k) 2^{-m} dt' \\ &= 2^{(m+l)/2} \int_{-\infty}^{\infty} \psi(t) \psi(2^{l-m}(t+n) - k) 2^{-m} dt \\ &= 2^{(m+l)/2} \int_{-\infty}^{\infty} \psi(t) \psi(2^{l-m}(t+n) - k) 2^{-m} dt \end{aligned}$$

Now when  $t < 0$  or  $t \geq 1$ ,  $\psi(t)=0$ , when  $0 \leq t < 1/2$ ,  $\psi(t)=1$ , and when  $1/2 \leq t < 1$   $\psi(t)=-1$ . Thus,

$$2^{(l-m)/2} \int_{-\infty}^{\infty} \psi(t) \psi(2^{l-m}(t+n)-k) dt = 2^{(l-m)/2} \int_0^{1/2} \psi(2^{l-m}(t+n)-k) dt - 2^{(l-m)/2} \int_{1/2}^1 \psi(2^{l-m}(t+n)-k) dt.$$

If  $l \neq m$ , the integrals are zero. When  $l=m$ , we have  $\int_0^{1/2} \psi(t+n-k) dt - \int_{1/2}^1 \psi(t+n-k) dt$ . If  $n \neq k$ , the expression is equal to zero. If  $n=k$ , we have  $\int_0^{1/2} \psi(t) dt - \int_{1/2}^1 \psi(t) dt = 1$ . Thus  $\psi_{mn}$  is orthonormal.

Since  $\psi_{mn}$  is a complete orthonormal basis of  $L^2(\mathbf{R})$ , for every  $f \in L^2(\mathbf{R})$ , we have the series

$$\bigoplus_{m,n=-\infty}^{\infty} c_{mn} \psi_{mn} = \bigoplus_{m,n=-\infty}^{\infty} \langle f, \psi_{mn}(t) \rangle \psi_{mn}(t) = f. \text{ If the approximate series begins from some } m, \text{ then the}$$

series can be written as:  $\bigoplus_{k,n=-\infty}^{m-1} c_{kn} \psi_{kn} + \bigoplus_{k=m}^{\infty} \bigoplus_{n=-\infty}^{\infty} c_{kn} \psi_{kn}$ . We define  $W_m = \bigoplus_{n=-\infty}^{\infty} c_{mn} \psi_{mn}$  for a fixed  $m$ . We

define  $V_m = \bigoplus_{k,n=-\infty}^m c_{kn} \psi_{kn}$ .  $V_{m+1} = W_{m+1} \oplus V_m$ , where  $\oplus$  denotes orthogonal union.

## 2. Multiresolutional Analysis

A multiresolutional analysis, denoted MRA, is a space denoted  $V_m$ , which is the span of  $\{\phi_{mn}\}$ . We define  $\phi$  as a scaling function whose translates form a complete orthogonal basis of  $V_0$  (span of  $\{\phi_{0n}\}$ , translates of  $\phi$  with no dilation). For multiresolutional analysis, we begin with a scaling function  $\phi$ , which is a real valued function on  $\mathbf{R}$  and generator of a multiresolutional analysis of  $L^2(\mathbf{R})$  as follows: (Given by Walter [10])

(i)  $\{\phi(t-n)\}$  is an orthonormal basis of  $V_0$

(ii)  $\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset L^2(\mathbf{R})$

(iii)  $f(t) \in V_m \Leftrightarrow f(2t) \in V_{m+1}$

(iv)  $\bigcap_m V_m = \{0\}$ ,  $\bigcup_m V_m = L^2(\mathbf{R})$

When there is a set of compactly supported functions which translate to form an orthogonal basis of  $V_0$  we say that  $(V_m)$  is an orthogonal multiresolutional analysis(MRA). If the functions which form  $V_0$  are continuous we say that  $(V_m)$  is a continuous multiresolutional analysis(MRA).

For univariate cases, when  $(V_m)$  is an orthogonal MRA it is possible to generate

compactly supported wavelets that create an orthogonal basis of  $W_0$ . To achieve this, one must find the orthogonal scaling functions.

### 3) Orthogonal Finite Shift Invariant Spaces

Let  $\Phi$  be a subset of  $L^2(\mathbf{R})$  and  $\tau(\Phi) = \{\phi(t-n) \mid n \in \mathbf{Z}, \phi \in \Phi\}$  give the set of integer translates of elements in  $\Phi$  and let  $\sigma(\Phi)$  be the  $L^2$  closure of the linear span of  $\tau(\Phi)$ . Then  $V \subset L^2(\mathbf{R})$  is a space called a finitely generated shift-invariant (denote FSI) space when  $V = \sigma(\Phi)$  for a finite set  $\Phi$ . If  $V_p$  is a MRA which is generated by  $r$  scaling functions, then  $V_p$  is a FSI space generated by  $2^p r$  generators (refer to [3]).

### 4) Macroelements and their Bezier Coefficients

#### 4.1) Definition of Bernstein Bezier coefficients.

We are able to define the functional value of a quadratic polynomial in terms of Bernstein Bezier coefficients (denote Bezier Coefficient). We start by defining the barycentric coordinate system. This coordinate system is defined over a triangle. We give the coordinate values of the three vertices  $V_0 = (1,0,0)$ ,  $V_1 = (0,1,0)$ , and  $V_2 = (0,0,1)$ . Any point lying within the triangle can have its coordinates expressed in terms of this coordinate system. The triangle is subtriangulated into three subtriangles by connecting the chosen point with each vertex (see Figure 4.1.1). Then the areas of the subtriangles,  $a_0$ ,  $a_1$ , and  $a_2$  are found. The coordinates of the chosen point are then given by  $(a_0/A, a_1/A, a_2/A)$ , where  $A$  is the area of the triangle with vertices  $V_0, V_1, V_2$ . This gives us the barycentric coordinate system. Thus we can express a point  $(x,y)$  in terms of barycentric coordinates as follows

$$(x,y) = u_1 V_0 + u_2 V_1 + u_3 V_2,$$

where  $u_1 + u_2 + u_3 = 1$ .



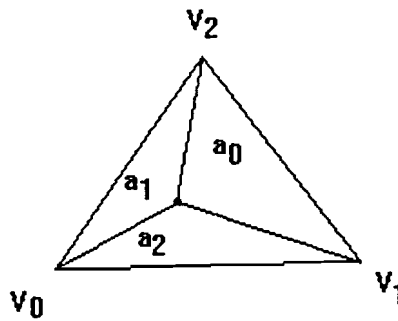


Figure 4.1.1

We can express a quadratic polynomial,  $P(x,y)$  in terms of its barycentric coordinates by use of what are called Bezier coefficients. We express

$$P(x,y) = \sum a_i (2!/i!) \mathbf{u}^i$$

where  $\mathbf{i} = (i_1, i_2, i_3)$ ,  $\mathbf{u} = (u_1, u_2, u_3)$ ,  $i! = i_1! i_2! i_3!$ ,  $\mathbf{u}^i = u_1^{i_1} u_2^{i_2} u_3^{i_3}$ , and  $a_i$  are called Bezier coefficients with respect to  $P(x,y)$ .

#### 4.2) Definition of Macroelements

A macroelement is a piecewise function defined over a triangle under some subtriangulation (triangular partition). Macroelements can be used to interpolate the given function values and two partial derivative values at the three vertices of the triangle.

#### 4.3) Definition of $C^1$ continuity

For a function to have  $C^1$  continuity, it must be differentiable all its first partial derivatives are continuous. Macroelements must be  $C^1$  continuous over the interior and the edge.

#### 4.4) Refinement of Macroelement

It is possible to use Bernstein-Bezier polynomials to generate smooth surfaces. Piecewise quadratic polynomials are the splines with the lowest possible degree and  $C^1$  continuity. Macroelements are splines defined on a triangular region  $T$ . There are two main steps for construction of  $C^1$  macroelements on  $T$ . First is to split the triangle into twelve subtriangular regions by using the three medians and line segments connecting the midpoints edges on  $T$ .

Next, to construct a quadratic element in each subtriangle under this division, use functional and first partial derivative values with respect to  $x$  and  $y$  at the three vertices of  $T$ . We denote this subtriangulation as the  $C^1$  quadratic twelve-subtriangular macroelement.

Let the triangle  $T = \langle V_1, V_2, V_3 \rangle$ , where  $V_i = (a_i, b_i)$ ,  $i = 1, 2, 3$ .  $T$  is divided into twelve subtriangles as shown in Figure 4.4.1, where the medians  $V_1V_{23}$ ,  $V_2V_{31}$ , and  $V_3V_{12}$  and the line segments  $V_{12}V_{23}$ ,  $V_{23}V_{31}$ , and  $V_{31}V_{12}$  are used in the subtriangulation. The medians are given by:

$$V_{12} = 1/2(V_1 + V_2), \quad V_{23} = 1/2(V_2 + V_3), \quad \text{and} \quad V_{31} = 1/2(V_3 + V_1).$$

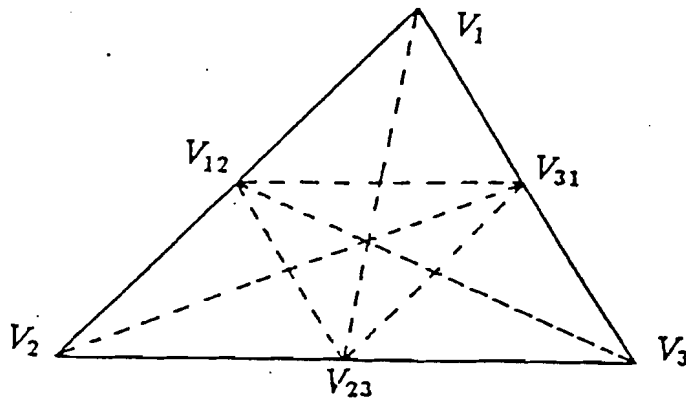


Figure 4.4.1

A quadratic polynomial defined on the triangle  $\langle V_1, V_2, V_3 \rangle$  can be expressed in terms of its Bezier coefficients  $a_i$ , and the barycentric coordinate system  $(x, y) = u_1A + u_2B + u_3C$  as:

$$P(x, y) = \sum a_i (2!/i!) \mathbf{u}^i, \quad \text{where } \mathbf{i} = (i_1, i_2, i_3), \quad \mathbf{u} = (u_1, u_2, u_3), \quad i! = i_1! i_2! i_3!, \quad \text{and } \mathbf{u}^i = u_1^{i_1} u_2^{i_2} u_3^{i_3}.$$

We will use the following parameters defined on  $T$ :  $d_i$ ,  $m_i$ ,  $n_i$ , and  $p_i$ . For  $i = 1, 2, 3$  use  $d_i$  to denote the function value at  $V_i$ ,  $m_i$  to denote the first partial derivative with respect to  $x$  at  $V_i$ , and  $n_i$  to denote the first partial derivative with respect to  $y$  at  $V_i$ . For  $I = 1, 2, 3$  define  $p_i$  as follows:

$$\begin{aligned} p_1 &= D_{k_1} f(V_{12}), \\ p_2 &= D_{k_2} f(V_{23}), \\ p_3 &= D_{k_3} f(V_{31}), \end{aligned}$$

with  $k_1 = (b_1 - b_2, a_2 - a_1)$ ,  $k_2 = (b_2 - b_3, a_3 - a_2)$ , and  $k_3 = (b_3 - b_1, a_1 - a_3)$ . For any vector  $k = (k_1, k_2)$ ,  $D_k f$  denotes the derivative of  $f$  with respect to  $k$  given by

$$D_k f = k \cdot \nabla f = k_1 (\delta f / \delta x) + k_2 (\delta f / \delta y).$$

Thus,  $p_1$ ,  $p_2$ , and  $p_3$  are normal derivatives at the midpoints of the edges  $V_1 V_2$ ,  $V_2 V_3$ , and  $V_3 V_1$  of  $T$ , respectively.

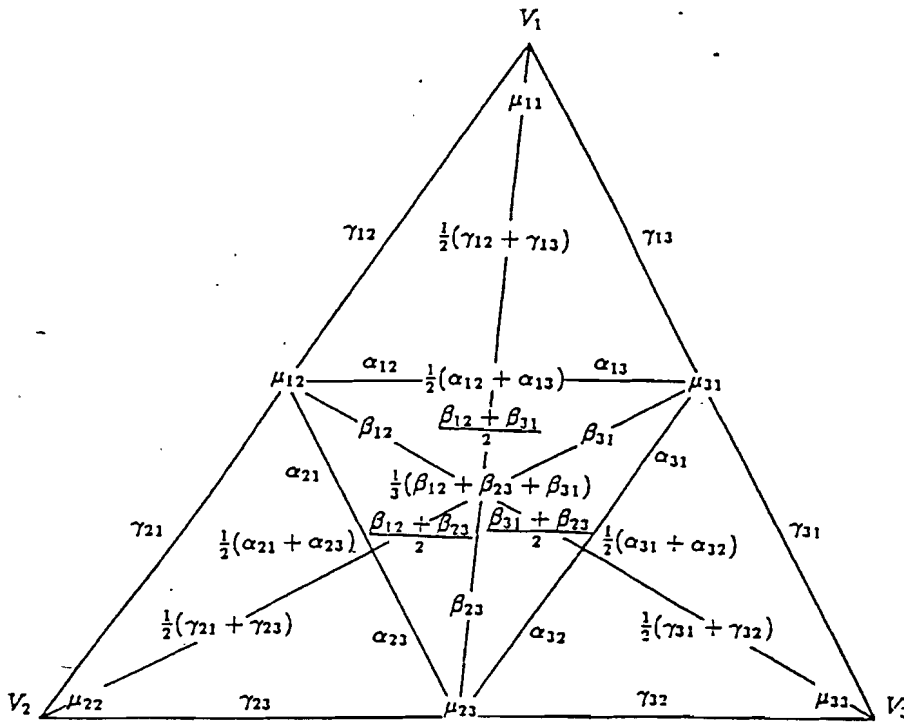


Figure 4.4.2

As shown in Figure 4.4.2, we can express all of the Bezier coefficients of the quadratic twelve subtriangulation macroelement on  $T$ . The respective Bernstein-Bezier coefficients shown in the subtriangulation are: (given by Chui and He [1]).

$$\mu_{ij} = (d_i + d_j)/2 + 1/8[(m_i - m_j)(a_j - a_i) + (n_i - n_j)(b_j - b_i)],$$

$$\gamma_{ij} = d_i + 1/4[m_i(a_j - a_i) + n_i(b_j - b_i)],$$

$$\alpha_{ij} = \mu_{ij} + 1/8[(a_k - a_j) \delta / \delta x f(V_{ij}) + (b_k - b_j) \delta / \delta y f(V_{ij})],$$

$$\beta_{ij} = \mu_{ij} + 1/24[(a_k - (a_i+a_j)/2) \delta/\delta x f(V_{ij}) + (b_k - (b_i+b_j)/2) \delta/\delta y f(V_{ij})]$$

Where  $i, j = 1, 2, 3$ , and  $k$  is the complement of  $\{i, j\}$  whenever  $i \neq j$ . The expressions necessary for finding the partial derivatives are:

$$\begin{aligned} (a_1-a_2) \delta/\delta x f(V_{12}) + (b_1-b_2) \delta/\delta y f(V_{12}) &= 2D_{v_1-v_{12}}f(V_{12}) \\ (b_1-b_2) \delta/\delta x f(V_{12}) - (a_1-a_2) \delta/\delta x f(V_{12}) &= p_1 \end{aligned}$$

$$\begin{aligned} (a_2-a_3) \delta/\delta x f(V_{23}) + (b_2-b_3) \delta/\delta x f(V_{23}) &= 2D_{v_2-v_{23}}f(V_{23}) \\ (b_2-b_3) \delta/\delta x f(V_{23}) - (a_2-a_3) \delta/\delta x f(V_{23}) &= p_2 \end{aligned}$$

$$\begin{aligned} (a_3-a_1) \delta/\delta x f(V_{31}) + (b_3-b_1) \delta/\delta x f(V_{31}) &= 2D_{v_3-v_{31}}f(V_{31}) \\ (b_3-b_1) \delta/\delta x f(V_{31}) - (a_3-a_1) \delta/\delta x f(V_{31}) &= p_3 \end{aligned}$$

## 5) Generating FSI over Simplexes

### 5.1) MRA in higher dimensions

For a given vector space in  $\mathbf{R}^n$  with  $e_1, \dots, e_n$  independent vectors, then a sequence of nested closed linear subspaces  $(V_m)$  in  $L^2(\mathbf{R})$  is a MRA of multiplicity  $r$  and dilation  $N$  if :

- 1)  $f \in V_m \Leftrightarrow f(N^{-m}t) \in V_0$  for  $N, m \in \mathbf{Z}, N > 1$
- 2)  $\bigcap V_m = \{0\}$
- 3)  $\bigcup V_m$  is dense in  $L^2(\mathbf{R})$
- 4) There exists a set of  $r$  functions,  $\phi_1, \dots, \phi_r$ , such that the set of lattice translates  $\{\phi_s(t-m_1e_1-\dots-m_n e_n) \mid s = 1, \dots, r \text{ and } m_1, \dots, m_n \in \mathbf{Z}\}$  form a complete orthogonal basis of  $V_0$  (Given by Donovan, Geronomo, and Hardin [2]).

To satisfy these conditions, it is necessary to have the set of scaling functions  $\phi_1, \dots, \phi_r$  generate the MRA. If there is a set of scaling functions whose lattice translates create an orthogonal basis of  $V_0$  we say that  $(V_m)$  is an orthogonal MRA.

To construct a set of scaling functions that satisfy the preceding properties, we have

derived a useful formula:

$$\langle P_n, Q_n \rangle_{\Delta S} = \iint_{\Delta S} P_n(x) Q_n(x) dx = S! V_{\Delta S} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (2n+s)! \prod_{k=1}^S \binom{ik+jk}{ik}, \quad S=2$$

where  $V_{\Delta S}$  is the volume of the s-dimensional simplex,  $\mathbf{i} = i_1+i_2+\dots+i_s$ ,  $\mathbf{j} = j_1+j_2+\dots+j_s$ , and  $\mathbf{a}_i$  and  $\mathbf{b}_j$  are respective Bernstein-Bezier coefficients of  $P_n(x)$  and  $Q_n(x)$ .

## 5.2) Derivation of Inner Product Formula

We have two piecewise polynomials  $P_n(x) = \sum \mathbf{a}_i \phi_i^n(\lambda)$  and  $Q_n(x) = \sum \mathbf{b}_j \phi_j^n(\lambda)$  where  $\phi_i^n(\lambda) = n!/i!$   $\lambda^i$ ,  $\phi_i^n(\lambda) = n!/i! \lambda^i$ ,  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ ,  $\mathbf{i} = i_1+i_2+i_3$ ,  $\mathbf{j} = j_1+j_2+j_3$ , and  $\mathbf{a}_i$  and  $\mathbf{b}_j$  are respective Bernstein-Bezier coefficients of  $P_n(x)$  and  $Q_n(x)$ .

$$\begin{aligned} \langle P_n, Q_n \rangle &= \iint_{\Delta} P_n(x) Q_n(x) dx dy \\ &= \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (i! j!) \iint \lambda_1^{i_1+j_1} \lambda_2^{i_2+j_2} \lambda_3^{i_3+j_3} dx dy, \end{aligned}$$

$$\text{where } dx dy = \begin{vmatrix} \delta x / \delta \lambda_1 & \delta x / \delta \lambda_2 \\ \delta y / \delta \lambda_1 & \delta y / \delta \lambda_2 \end{vmatrix} \begin{vmatrix} \delta \lambda_1 & \delta \lambda_2 \\ \delta \lambda_1 & \delta \lambda_2 \end{vmatrix} = \begin{vmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{vmatrix} \begin{vmatrix} \delta \lambda_1 & \delta \lambda_2 \\ \delta \lambda_1 & \delta \lambda_2 \end{vmatrix} = V_{\Delta} \delta \lambda_1 \delta \lambda_2.$$

$$\begin{aligned} &\sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (i! j!) \iint_{\Delta} \lambda_1^{i_1+j_1} \lambda_2^{i_2+j_2} \lambda_3^{i_3+j_3} dx dy \\ &= V_{\Delta} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (i! j!) \int_0^1 \int_0^{1-\lambda_1} \lambda_1^{i_1+j_1} \lambda_2^{i_2+j_2} (1-\lambda_1-\lambda_2)^{2n-(i_1+j_1)-(i_2+j_2)} d\lambda_2 d\lambda_1. \end{aligned}$$

Since,

$$\int_0^{1-\lambda_1} \lambda_2^{(i_2+j_2+1)-1} (1-\lambda_1-\lambda_2)^{[2n-(i_1+j_1)-(i_2+j_2)+1]-1} d\lambda_2 = (1-\lambda_1)^{[2n-(i_1+j_1)+2]-1} B(i_2+j_2+1, 2n-(i_1+j_1)-(i_2+j_2)+1),$$

we have

$$\begin{aligned} &V_{\Delta} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (i! j!) \int_0^1 \int_0^{1-\lambda_1} \lambda_1^{i_1+j_1} \lambda_2^{i_2+j_2} (1-\lambda_1-\lambda_2)^{2n-(i_1+j_1)-(i_2+j_2)} d\lambda_2 d\lambda_1 \\ &= 2V_{\Delta} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j B(i_2+j_2+1, 2n-(i_1+j_1)-(i_2+j_2)+1) [(n!)^2 / (i! j!)] \int_0^1 \lambda_1^{i_1+j_1} (1-\lambda_1-\lambda_2)^{2n-(i_1+j_1)-(i_2+j_2)} d\lambda_1 \\ &= 2V_{\Delta} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j B(i_2+j_2+1, 2n-(i_1+j_1)-(i_2+j_2)+1) [(n!)^2 / (i! j!)] B(i_1+j_1+1, 2n-(i_1+j_1)+2) \\ &= 2V_{\Delta} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j [(n!)^2 / (i! j!)] [((i_2+j_2)! (2n-(i_1+j_1)+1)! / (2n+2)!)] [((i_1+j_1)! (i_3+j_3)! / (2n-(i_1+j_1)+1)!)] \\ &= \iint_{\Delta S} P_n(x) Q_n(x) dx = S! V_{\Delta S} \sum_i \sum_j \mathbf{a}_i \mathbf{b}_j (n!)^2 / (2n+s)! \prod_{k=1}^S \binom{ik+jk}{ik}, \end{aligned}$$

where  $s=2$ .

### 5.3) Application of Inner Product Formula In generating MRA.

Using an example given by Donovan, Geronomo, and Hardin (refer to [3]), we can rework the example using our derived formula for inner products. Let  $\phi_{k+1}, \dots, \phi_r$  be spanned by the function  $g_1$  and  $\phi_1, \dots, \phi_k$  be spanned by  $g_0$ . We find that  $\langle g_0, g_1 \rangle \neq 0$ , thus we must introduce a scaling function  $\omega$  such that  $g_0$  and  $g_1$  are orthogonal i.e.

$$\langle g_0, g_1 \rangle = \langle g_0, \omega \rangle^2 / \langle \omega, \omega \rangle = 0$$

This above formula was given by Donovan, Geronomo, and Hardin as means of generating scaling functions necessary for a two dimensional MRA.

The formula for computation of  $\langle P_n(x), Q_n(x) \rangle_{\Delta_S}$  is utilized to provide the respective inner products.

Consider the equilateral lattice  $L$  in  $\mathbf{R}^2$  generated by vectors  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (1/2, \sqrt{3}/2)$ . We let  $\Delta$  denote the triangle with vertices  $(0, \mathbf{e}_1, \mathbf{e}_2)$  and  $\Delta'$  denote the triangle with vertices  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)$ . We let  $r$  be the reflection across the common edge of the triangles. We allow  $T$  to be the triangulation that consists of  $\Delta$  and  $\Delta'$ . We define  $h$  to be the hat function of the lattice, that is  $h$  is a piecewise linear function on  $T$  and  $h(i\mathbf{e}_1 + j\mathbf{e}_2) = \delta_{i,0} \delta_{j,0}$ . An orthogonal MRA is generated from  $h$ .

We have that  $\omega$  is expressed as a linear combination of  $h \bullet u_{ij}$ ,  $\omega \bullet u_{ij}$ , and  $\omega \bullet r \bullet u_{ij}$  where  $u_{ij}(x) = 3x - i\mathbf{e}_1 - j\mathbf{e}_2$  and  $\tau_{ij} = x - i\mathbf{e}_1 - j\mathbf{e}_2$  for all  $x \in \mathbf{R}^2$ .

We let  $g_0 = hX_\Delta$ ,  $g_1 = h \bullet \tau_{1,0} X_\Delta$ , and  $g_2 = h \bullet \tau_{0,1} X_\Delta$ , where  $X_\Delta$  denotes the characteristic function on  $\Delta$ . We can compute the following inner products:

$$\begin{aligned} \langle h, 1 \rangle_\Delta &= 2A \sum_{i,j} \mathbf{a}_i \mathbf{b}_j (n!)^2 / (2n+2)! \binom{i_1 + j_1}{i_1} \binom{i_2 + j_2}{i_2} \binom{i_3 + j_3}{i_3} \\ &= 2(\sqrt{3}/2) \sum \mathbf{a}_i (1/4!) \binom{i_1 + 1}{i_1} \binom{i_2 + 1}{i_2} \binom{i_3 + 1}{i_3} \end{aligned}$$

$$\begin{aligned}
&= 2(\sqrt{3}/2)(1/4!) \sum_{1,2,3} \mathbf{a}_i \binom{1+i}{i} \binom{0+i}{0} \binom{0+i}{0} \\
&= 2(\sqrt{3}/2)(1/4!)(1+0+0)(2)(1)(1) \\
&= \sqrt{3}/12
\end{aligned}$$

and,

$$\begin{aligned}
\langle \mathbf{h}, \mathbf{h} \rangle_{\Delta} &= 2A \sum_{ij} \mathbf{a}_i \mathbf{b}_j (n!)^2 / (2n+2)! \binom{i_1+j_1}{i_1} \binom{i_2+j_2}{i_2} \binom{i_3+j_3}{i_3} \\
&= 2(\sqrt{3}/4)(1/4!) \sum_{1,2,3} \mathbf{a}_i [1] \\
&= (\sqrt{3}/2)(1/4!)1 \\
&= \sqrt{3}/24.
\end{aligned}$$

Hence,  $\langle \mathbf{h}, \mathbf{1} \rangle = \sqrt{3}/2$  and  $\langle \mathbf{h}, \mathbf{h} \rangle = \sqrt{3}/4$ . Likewise  $\langle \mathbf{g}_0, \mathbf{g}_1 \rangle = \sqrt{3}/48$  can be shown.

Because  $\omega$  satisfies a nonhomogeneous dilation equation, it can be expressed in the form:

$$\omega = \mathbf{h} \bullet \mathbf{u}_{1,1} + \sum_{(i,j) \in Q} s_{ij} \omega \bullet \mathbf{u}_{ij} + \sum_{(i,j) \in Q_r} s'_{ij} \omega \bullet \mathbf{r} \bullet \mathbf{u}_{ij}$$

where  $Q = \{(0,0), (0,1), (1,0), (1,1), (0,2), (2,0)\}$  and  $Q_r = \{(0,0), (1,0), (0,1)\}$ .

For  $\omega$  to be continuous,  $|s_{ij}| < 1$  for  $(i,j) \in Q$  and  $|s'_{ij}| < 1$  for  $(i,j) \in Q_r$ . Also because  $\omega$  must be symmetric with respect to rotation leaving  $\Delta$  fixed, we must have:

$$\begin{aligned}
s_{0,0} &= s_{2,0} = s_{0,2} = s_1 \\
s_{0,1} &= s_{1,0} = s_{1,1} = s_2 \\
s'_{0,0} &= s'_{1,0} = s'_{0,1} = s_3.
\end{aligned}$$

$$\begin{aligned}
\langle \omega, \mathbf{1} \rangle &= \langle \mathbf{h} \bullet \mathbf{u}_{1,1}, \mathbf{1} \rangle + \sum_Q s_{ij} \langle \omega \bullet \mathbf{u}_{ij}, \mathbf{1} \rangle + \sum_{Q_r} s'_{ij} \langle \omega \bullet \mathbf{r} \bullet \mathbf{u}_{ij}, \mathbf{1} \rangle \\
&= 1/9 \langle \mathbf{h}, \mathbf{1} \rangle + \sum_Q s_{ij} \langle \omega, \mathbf{1} \rangle + \sum_{Q_r} s'_{ij} \langle \omega, \mathbf{1} \rangle \\
&= (1/9) \langle \mathbf{h}, \mathbf{1} \rangle + (1/3) \left( \sum_{i=1}^3 s_i \right) \langle \omega, \mathbf{1} \rangle.
\end{aligned}$$

Solving for  $\langle \omega, \mathbf{1} \rangle$  yields:  $\langle \omega, \mathbf{1} \rangle = (1/3) \langle \mathbf{h}, \mathbf{1} \rangle / (3 - \sum_{i=1}^3 s_i)$ .

Since  $X_{\Delta} = \mathbf{g}_0 + \mathbf{g}_1 + \mathbf{g}_2$ , we have:  $\langle \omega, \mathbf{g}_0 \rangle = (1/3) \langle \omega, \mathbf{1} \rangle$ .

$\langle \omega, \omega \rangle$  is calculated in a similar fashion:

$$\begin{aligned}
\langle \omega, \omega \rangle &= \langle h \bullet u_{1,1}, h \bullet u_{1,1} \rangle + 2 \sum_Q s_{ij} \langle \omega \bullet u_{ij}, h \bullet u_{1,1} \rangle + 2 \sum_{Qr} s'_{ij} \langle \omega \bullet r \bullet u_{ij}, h \bullet u_{1,1} \rangle \\
&\quad + \sum_Q (s_{ij})^2 \langle \omega \bullet u_{ij}, \omega \bullet u_{ij} \rangle + \sum_{Qr} (s'_{ij})^2 \langle \omega \bullet r \bullet u_{ij}, \omega \bullet r \bullet u_{ij} \rangle \\
&= 1/9 (\langle h, h \rangle + 2 \sum s_{ij} \langle \omega, h \bullet \tau_{1-i,1-j} \rangle + 2 \sum s'_{ij} \langle \omega \bullet r, h \bullet \tau_{1-i,1-j} \rangle + \sum (s_{ij})^2 \langle \omega, \omega \rangle + \sum (s'_{ij})^2 \langle \omega, \omega \rangle) \\
&= (1/9) (\langle h, h \rangle + 6s_2 \langle \omega, g_0 \rangle + 6s_3 \langle \omega, g_0 \rangle + 3 \sum_{i=1}^3 (s_i)^2 \langle \omega, \omega \rangle).
\end{aligned}$$

$$\langle \omega, \omega \rangle = ((1/3) \langle h, h \rangle + 2 (s_2 + s_3) \langle \omega, g_0 \rangle) / (3 - \sum_{i=1}^3 (s_i)^2).$$

The equation from above, given by Donovan, Geronomo, and Hardin

$$\langle g_0, g_1 \rangle = \langle g_0, \omega \rangle^2 / \langle \omega, \omega \rangle$$

can be satisfied.

$$\begin{aligned}
\langle g_0, \omega \rangle^2 / \langle \omega, \omega \rangle &= -33 + 54s_1^2 - 25s_1^2 + 18s_2 - 6s_1s_2 - 13s_2^2 + 18s_3 - 6s_1s_3 + 6s_2s_3 - 13s_3^2 \\
&= 0.
\end{aligned}$$

For  $\omega$  to be continuous,  $|s_i| < 1$ . Values can be found where  $s_1, s_2,$  and  $s_3$  fulfill the above stated criteria. The values of the  $s_i, I=1,2,3$  are used to find an  $\omega$  such orthogonal scaling functions,  $\phi^1, \phi^2, \phi^3$ , can then be found.

### Acknowledgments

I would like to acknowledge and thank Dr. N. Jaggi, Dr. M. Jeter, and Dr. R. Sanders for their time in overseeing my work and suggestions made. I would also like to thank the members of my committee (above mentioned) for the instruction that I have received inside and outside of the classroom, without which I would be not have the background or understanding necessary for this research. Finally, I would like to thank Dr. T. X. He for being my faculty advisor on this project and for all the time and instruction which he has given to me.

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For further discussion of barycentric coordinates and Bezier coefficients, the reader is referred to [4, chapters 8, 18]

For further discussion of  $C^1$  continuity, the reader is referred to [4, chap7.3]

For further discussion of Bessel's inequality, the reader is referred to [5, page 150] and [9, pages 173 and 252]